Classroom discourse that affects reification of a mathematical object: The case of function

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Introduction

This research aims at constituting classroom discourse that affects reification of a mathematical object. Reification is a transition from operational procedure into a mental object. Researchers and a secondary teacher collaboratively designed a teaching unit on linear functions in one Japanese 8th-grade classroom.

Based on research about reification of mathematical objects (Font et.al, 2010; Nachilieli & Tabach, 2011; Nunokawa, 2014; Sfard, 2011) and design research (Gravemeijer & Cobb, 2006), the team designed a teaching unit to encourage students to think and talk about linear functions as reified objects in lessons. The teaching unit includes a milieu of coherent discourse and a dynamic ICT environment.

Rationale and related research

In our current course of study in secondary mathematics, 'function' is introduced as one of the four content areas (MEXT, 2010). For functions, the curriculum aims to investigate properties of functions by representing change and correspondence of two co-varying quantities as numerical tables, algebraic expressions, and graphs. However, understanding functions is difficult for junior secondary students. According to the national survey of achievement by National Institute for Educational Policy Research (NIER, 2012), one of the problems to be resolved is to develop students' understanding relations between two quantities as direct, inverse, and linear functions (p. 27).

Nunokawa (2014) undertook a comparative analysis of how Japanese textbooks introduce a definition of function and how it is described throughout the unit. There was substantial inconsistency in statements about functions, and this can cause unintentional difficulties for students in understanding function as a reified mathematical object represented by tables, algebraic expressions, and graphs. Font et al. (2010) found similar inconsistencies in classroom discourse about functions, for example: function is a one-to-one correspondence between two variables; a linear function $y=2x+1$; a function is represented by an algebraic expression. The second example identifies a function *with* an algebraic expression but the last one distinguishes a function *from* an algebraic expression. This illustrates a possible inconsistency in classroom discourse. Ensuring consistency in the classroom discourse about functions might enable a student to participate in learning functions in a more sensible way and might also affect reification of function as a mathematical object. In this regard, we would like to review recent research in classroom discourse.

Classroom discourse

According to Sfard (2011), discourse is a specific type of communication, which has interrelated characteristic features: special keywords; visual mediators; distinctive routines; and generally endorsed narratives (p. 2). Mathematics is conceived as a discourse that creates its own mathematical objects and students would reify mathematical objects by participating in mathematical discourse in which they talk about an object and the existence or nature of it. However, Nachilieli & Tabach (2011) analyzed 7th grade students as they were making their first steps in the discourse of functions for nearly two months. Their analysis showed that the students were able to participate in the discourse without ever dealing directly with this as-yet nonexistent object. Thus, the students managed to cope with problems by associating them with solution routines through all kinds of discursive clues to which they were sensitized through their former experience. Classroom function discourse was characterized as 'lower' level (Nachilieli & Tabach, 2011, p 25) - based on concrete calculation and operations on quantities given in a table, expression, or graph. In order to support students in reifying function as a mathematical object it is crucial for the teacher to identify a leading discourse, which guides students to proceed from the talk on function as 'inter-discursive' or 'discourse-for-others' to 'intra-discursive' or 'object-driven' (Nachilieli & Tabach, 2011, p. 25-26). Therefore we set out to design a coherent teaching unit for students to purposefully investigate co-varying quantity, represent its properties, and talk about functions as existing objects.

Design principles

We adopt the following three heuristics as design principles:

Leading discourse on function

We made a distinction between function and its tabular, algebraic, and graphical representation. Further, we adopt a definition of function not as an abstract one-to-one correspondence, but as a concrete quantity. This is the definition that was formulated by Euler (1775/2000) in his textbook on differential calculus:

> Hence, if x designates the variable quantity, all other quantities that in any way depend on x or determined by it are called its function. (p. vi)

We conceive numerical tables, algebraic expressions, and graphs as traces or shadows of the function itself. We expect students to grasp properties of particular functions from bits and pieces of the shadows.

Observing and telling properties of changing quantity in dynamic ICT environment

In order to emphasize function as a concrete quantity that depends on or is determined by a given quantity, the research design made use of the dynamic mathematical software 'GeoGebra.' We expect students to observe the changing quantity while manipulating a tablet PC with GeoGebra applets. The GeoGebra applets were developed by Nunokawa.

The Applet 'Counter' exhibits a numerical expression in which independent and dependent variables change simultaneously. The Applet 'Distinction' exhibits both a cartoon character and its shadow moving simultaneously on a coordinate plane, which suggests a natural distinction between a function and its graphic representation. In addition, the dynamic environment allows students to observe a qualitative aspect of 'rate of change' for a linear function and also for inverse proportion on the coordinate plane. In contrast to a linear function, the rate of change of inverse proportion is not constant and varies dramatically as x comes close to the origin. Further, it is expected that working with a tablet PC in a dynamic environment encourages students to have more opportunities for making conjectures and talking about properties of the varying dependent quantity.

Algebraic and geometric treatments of function

In all Japanese eighth grade textbooks, linear function is defined by the algebraic expression '$y=ax+b$' and is presented as a static mathematical object: an algebraic expression of a graph that contains two points; a common point of two graphs by solving a system of equations in two unknowns. Graphs of linear functions are recognized as straight lines, and relations between them are described by geometrical properties.

Premature introduction of linear functions as static objects, and treatment of both algebraic and geometrical discourses on linear function causes inconsistency with the characterization of function as a concrete quantity that depends on or is determined by a given quantity. In this experiment, we engineer a progressive change of discourse from the dynamic aspect of co-varying quantities towards the static aspects of reified objects.

Designing a unit and a milieu

The team of researchers and expert teacher collaboratively designed a teaching unit and milieu, which contains the intended leading discourse and the ICT environment.

Telling function: anthropomorphize function as Japanese "Ninja"

Based on the research on how the role of discourse affects reification of mathematical objects, we decided to distinguish function from its numerical, algebraic, and graphical representation and to talk of function coherently as the varying quantity y that is somehow determined by x. Further, in accordance to Hino's suggestion, we decided to anthropomorphize a function as a Japanese "Ninja" and to conceive of an algebraic expression, a numerical table, and a graph as shadows of a "Ninja" who gave a glimpse of its existence. The traditional image of Ninja is associated with stealth and invisibility. We expected the existence of Ninja and its images to help students to reify function as object and to conceive of its representations as shadows of the object.

In the lesson, the teacher and the students presupposed that there are many kinds of function that are named after Ninja: Ninja of 'direct proportion,' 'inverse,' and 'linear function' and so forth. Every Ninja will have a particular move; different Ninja will move differently. Though a function itself is stealthy and invisible, we can detect properties of change of the function and discern two linear functions from a glimpse of shadows.

In the course of the lesson, we expect the students to have some sense of the virtual existence of functions. Further, we expect the students to say something like: 'the inverse proportion Ninja moves at different paces. Indeed, the Ninja moves much more slower when far from the origin of the coordinate plane' or 'this linear function Ninja moves faster than this linear function Ninja.'

Observing quantity changing in dynamic ICT environment: GeoGebra applets

We designed several kinds of GeoGebra applets for the teacher's demonstrations and students' observations. For example, in order to find an intersection point of two straight lines that represent the respective linear function, we designed an applet in which two Ninja move on the *y*-axis at different paces, at the same time, two corresponding cumulative shadows appear on the coordinate plane (see Figure 1). With this applet, 'two Ninjas exchange secret messages,' we ask the students to find the meeting point and time in the coordinate system by applying properties of graphs of linear functions. The students are expected to make the distinction between a function and its graphic representation easily. Hence, the applet supports the student to find the coordinates using the idea of 'rate of change' instead of the algebraic equation, thus retaining a consistent discourse of function as co-varying quantities.

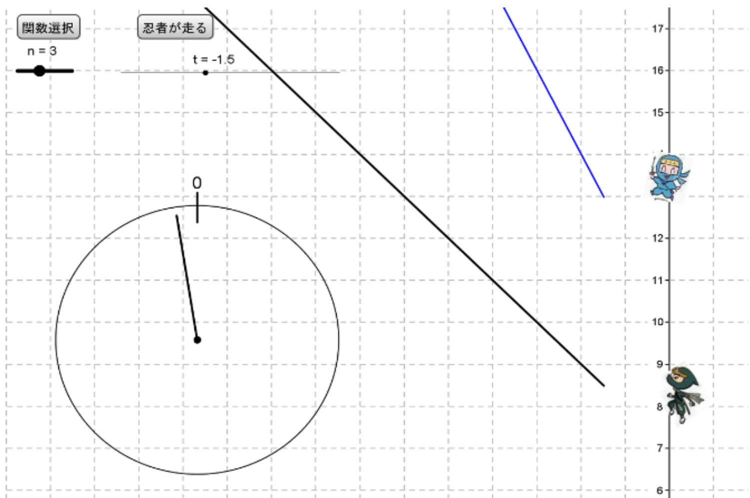

Figure 1. Two Ninjas exchange secret messages

Introduction of discourse on function as algebraic approach: "Ninjutsu"

In addition to finding the intersection of two straight lines, there are tasks that ask for finding algebraic expressions of a linear function given the coordinates of one point and gradient of graph, or two points. We extended the quantity approach to these tasks, which ask students to draw graphs and apply the property of rate (or pace) of change to find the parameters *a* and *b* in $y=ax+b$. Then, we asked students for a solution without graphing. Hence, we introduced discourse on function as an algebraic approach associated with the Ninja instruction: 'Find answer using 'Calculation Ninjutsu.''. By 'Ninjutsu' we mean a specific magical strategy used by Ninja. This algebraic approach is effective when the coordinates of the intersection point of two straight lines consist of rational values. We recommended that students 'master Ninjutsu' through apprenticeship.

Unit plan

Members of this research team decided on the baseline of the unit plan together. Kanno developed worksheets and Nunokawa, the GeoGebra applets. The team discussed again the balance in the use of applets between the teacher's demonstrations and students' hands-on observations. Table 1 shows the outline unit plan.

Table 1. A teaching unit plan

Sub-unit	Day	Topic
1	1	Direct and inverse proportion. Two perspectives of functions: “change” and “correspondence”
	2	The Ninja is running. To recognise features of “correspondence” in tabular, algebraic and graphical representations.
	3	To know the features of “change” look in tabular, algebraic and graphical representations.
	4	In a water tank situation, introduction to Ninja of linear function by examining the features of change different from those of Ninjas of direct and inverse proportion. To know that its pace of change is constant and to know how it is expressed and what it looks like in 'shadow' representations.
	5	Looking at the pace of change of inverse proportion. To know that the “rate of change” is the key to knowing the features of change.
	6	Examining the situation of incense stick burning. This is another Ninja of linear function. There are many Ninjas who move differently. The students create their names.
2	7	Review of first semester lessons. To find the initial length of incense stick from table, graph and algebraic expression. Extend number of Ninjas of linear function.
	8	Think about discerning Ninjas from algebraic expression 'shadows'. Pay attention to constant values in the expression and discern Ninjas from those values.
	9	Think about discerning Ninjas from graphs. Discern Ninjas with naïve expressions such as “point that passes through (y-intercept)” and “direction (gradient).”
	10	Find relationships between graphical and algenraic information to discern the Ninjas.
	11	Draw a graph from an algebraic expression and vice versa.
	12	To determine the algebraic expression from the y-intercept and gradient of the graph. Find the algebraic expression from an arbitrary point and gradient.
	13	Determine the algebraic expression from the two points that are left in a shadow of graph. Find the algebraic expression from arbitrary two points.
	14	Determine algebraic expression by “Calculation Ninjutsu” without using graph.
	15	Determine the intersection point of two straight lines by using “Calculation Ninjutsu.”
	16	Master “Calculation Ninjutsu.”
3	17	Assume the rise of temperature of water is a linear function and use interpolation/extrapolation.
	18	Compare telephone price rates between two companies.
	19	Explore a walking path by Ms. Ikeda who leaves home and visits a shop on the way to visit an acquaintance.

Design experiment and results

Ms. Kanno with whom we have collaborated over 17 years, implemented the teaching experiment on linear function from July 3 to October 20, 2014. She taught all the lessons for eleven lower achieving students.

Data collection and analysis

Every lesson, three times a week, was video recorded for later analysis. The private utterances and written work of four students were video recorded by a micro camera fixed on their desk. Written documents were also collected and scanned.

Illustration of classroom discourse that affects reification of mathematical objects

Preliminary analysis of the entire corpus of data generates a provisional typography of discourse, characterized by level (higher/lower) and referent (object/non-object).

As preliminary results, we found discourse that refers to a function itself apparent in the second sub-unit in which function was personified as Ninja and its representations characterized as shadows. We conceived the discourse that affects objectification of function to be at the higher level and referring to specific characteristics of co-varying quantities such as 'rate of change.' Such aspects of discourse were discerned in sub-unit two. In the following, we illustrate this from days 5 and 9.

Day 5: Students compare 'rates of change' of linear function and inverse proportion. The lower discourse level was dominant for the first part of the lesson when the students calculated rate of change with given numerical data. Higher and object-reference discourse occurred while the students observed a GeoGebra applet which shows rate of change dynamically. They observed that for inverse proportion 'pace varies,' 'at first, the pace suddenly declines,' 'rate of decline goes gradually smaller' and so on. These utterances refer to specific characteristics of varying quantity and thus are categorized as object–oriented discourse on function.

Day 9: Teacher demonstrated moving Ninjas along the y-axis and the corresponding graphs. Students themselves operated this with a tablet PC. We could not detect higher and object-referred discourse in the whole classroom episode. However, one of the selected students (TM) described a characteristic of co-varying quantities in his work sheet (See Figure 2). TM invented an idea of quantity changing of two Ninja in terms of 'degree and speed.' The students were asked to 'write down your method of distinction between the shadowgraphs 3 and 4' (these are represented in $y=2x+2$ and $y=x+2$ respectively). TM wrote that '④ about 80°and the speed is fast.', '⑤ about 60°and compared with 4, the speed is slow.' Since the idea of speed has nothing to do with the graph, we can conclude that TM is describing changing quantity.

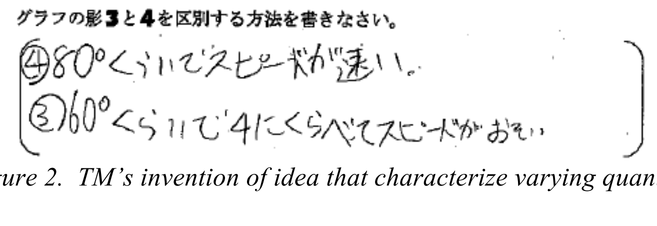

Figure 2. TM's invention of idea that characterize varying quantity

Conclusion

By setting appropriate contexts and using dynamic and interactive representations, we directed students' attention to features of changes of variables and succeeded in promoting discussions whose main topics were those features. Since changes of variables are critical features of functions, it can be said that our unit could construct a discourse where functions were the objects of talk, even for these low attaining students.

We had not asked the teacher to intentionally direct the students' attention to changes of variables in the latter part of the unit, because we expected this focus to be maintained. Contrary to our expectation, the class did not mention changes of variables and examined only surface relationships between those expressions and graphs. This suggests that the intentional construction of the discourse through explicit mention of changes of variables should have been maintained even in the latter part of the unit.

Nachlieli & Tabach (2011) claim: "students should have probably spent more time getting acquainted with the three lower-level discourses before the subsuming discourse on function was introduced" (p. 25). Our research provides an alternative route: highlighting changes of variables and encouraging students to talk about them. A continuation of this idea throughout the whole unit may improve effectiveness.

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