Past and future of research on mathematical reading related to proofs

Kai-Lin Yang, National Taiwan Normal University, Taiwan

Linguistic nature of mathematics

The language of mathematics can be regarded as the inclusion of ordinary language and symbolic language that can represent patterns underlying the physical or the imaginary world. Although mathematics does not equal to the language of mathematics, one core of comprehending and constructing mathematical relationships lies in the utilization and development of the language of mathematics. Much of the difficulty in learning mathematics has to do with the language of mathematics (Thurston, 1994). The learning of mathematics is different from the learning of ordinary language. According to Piaget's (1951) perspective on the child's intellectual development, the first stage of development is the sensorimotor stage in which thought begins the development through participation in activities with communicative others, and then interactive activities constantly build the language of experiences. From this perspective, the learning of mathematics emanates from learners' mathematical experiences and mathematical language provides a means of representing what is known. Thus, the nature of mathematics makes the learning different from the learning of ordinal language. According to Vygotsky's (1962) perspective, social experiences make the crucial contribution to the process of developmental change. Moreover, thought and language are not inner and outer manifestations of the same phenomenon, but two distinct cognitive operations that are interdependent. Thus, I notice the multi-semiotic transformations of mathematics, which go beyond what ordinary language can express, e.g. modes of reasoning and the relational dense noun phrases (Radford, 2003; Schleppegrell, 2010). Based on the linguistic nature of mathematics, mathematical reading is functionally different from reading in other subjects. In particular, proofs are viewed as a special genre of mathematical texts, and reading is one approach to understanding proofs (Yang, 2012).

In the following sections, I first discuss the previous work on mathematical reading related to proofs according to three key components in reading – the text, the reader, and the context. Then, I look back at the work based on multiple meanings of mathematical reading for further research on mathematical reading.

Three components in research on mathematical reading related to proofs

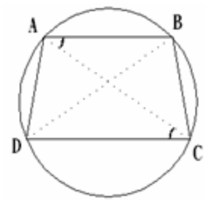
The reader, the text and the context are three key components involved in reading (Rosenblatt, 1978; Rumelhart, 1994). For secondary literacy researchers, the reader has more central roles in the process of constructing reading (Tierney & Pearson, 1992). What counts as text can refer to both the original text and a new text emerging from the evolution of meanings in reading (Bloome & Egan-Robertson, 1993). The context is more diverse to shape multiple purposes and various meanings generated from reading (Moje, Dillon & O'Brien, 2000). In view of the importance of the three components, I structure the review of current research on mathematical reading based on the three components.

The Text

In learning mathematical ideas through reading, we need to pay much more attention to not just definitions of mathematical terms and the meanings of symbols, but also grammatical structures of mathematical texts and the mathematical thoughts underlying mathematical texts (Morgan, 1996). Thus, the content of reading comprehension is the first dimension concerned in the component of the text. As to the dimension, mathematics education researchers have revealed that students' reading comprehension of mathematics proof is complex and called for developing comprehensive frameworks for assessing students' ability to learn mathematics by reading (Conradie & Frith, 2000; Mejia-Ramos et al., 2012; Yang & Lin, 2008). Following Yang & Lin (2008), reading comprehension of proofs means to understand a proof from basic knowledge of mathematical terms and symbols in this proof, the essential elements of how this proof operates and why this proof is right, to what this proof can validate and the application of the proposition validated by this proof. In their study, students' reading comprehension of geometry proof (RCGP) can be evaluated according to their performance on the five facets, basic knowledge, logical status, summarization, generality and application. The five facets of RCGP were not only corresponding to Bloom's taxonomy for further elaboration, but also structured into four comprehension levels – surface, recognising elements, chaining elements, and encapsulation – which had been confirmed by students' performance (Yang & Lin, 2008).

The second dimension of the text is variables, which can be considered to design the text for better comprehension. This dimension can be viewed as the pedagogical function of the text. While proof is viewed as a genre of mathematical text, it is expected that one rethink the features of the text for benefiting students' RCGP. It is difficult for students to produce a proof in the two-column format, but this format may benefit students' RCGP. In proofs in the two-column format, proof steps were separated into conclusion at the left and reason at the right, as shown in Figure 1.

As figure shown, A, B, C and D lie on the same circle. Prove that ABCD is an isosceles trapezium if Arc (AD) = Arc (BC).



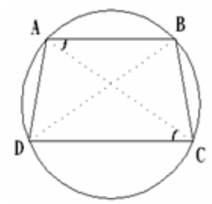
Proof:

Conclusion：	Reason：
1. $\angle BAC = \angle ACD$ and $\overline{AD} = \overline{BC}$.	1. Arc (BC) = Arc (AD)
2. $\overline{AB} // \overline{DC}$	2. $\angle BAC = \angle ACD$ and the interior alternate angles are equal
3. $ABCD$ is an isosceles trapezium	3. $\overline{AB} // \overline{DC}$ and $\overline{AD} = \overline{BC}$

Figure 1. A proof in the two-column format.

In the past two decades, most school textbooks present proofs in line-by-line formats, as shown in Figure 2. Yang, Lin and Wang (2008) justified whether the two-column and line-by-line formats influence students' RCGP and found the written formats did not result in statistically significant difference of students' RCGP. Further studies are necessary to know more about how students read proofs in each format for explaining why there is no difference. In addition, Chen and Yang (2010) considered the variable of the text structure as to "Pythagorean Theorem with its proof and its application," and justified whether the proof-first or application-first text is better for students' understanding and application of this theorem. They found students of the lowest one-third pretest scores performed better on procedural knowledge and problem solving when reading the application-first text.

As figure shown, A, B, C and D lie on the same circle. Prove that ABCD is an isosceles trapezium if Arc(AD) = Arc(BC).



Proof:

Because Arc (BC) = Arc (AD), $\angle BAC = \angle ACD$ and $\overline{AD} = \overline{BC}$.

Because $\angle BAC = \angle ACD$, $\overline{AB} // \overline{DC}$ (the interior alternate angles are equal).

Because $\overline{AB} // \overline{DC}$ and $\overline{AD} = \overline{BC}$, ABCD is an isosceles trapezium.

Figure 2. A proof in the line-by-line format

The third dimension of the text is related to mathematicians' practice of reading. How mathematicians read the text is of interest for pedagogical implication. Yang and Lin (2008) interviewed five mathematicians (two in the field of number theory, one in analysis, one in statistics, and one in geometry) to learn about how they read mathematical proofs. They found that mathematicians particularly paid attention to the connection between theorems or proofs, the improvement of proof methods, the value and the beauty of a theorem and its proof. Their reading comprehension of proofs could be divided into two stages. One is to read for oneself during the process of conjecturing propositions and germinating proof ideas, and the other is to read for others during the process of reorganizing the structure and writing for publication.

The reader

Three dimensions are identified in relation to the reader – mathematical knowledge, logical reasoning and reading strategies from past studies. As to students' prior knowledge and logical reasoning, Lin and Yang (2007) found that both *logical reasoning* and *relevant geometry knowledge* were not only moderately correlated with RCGP, but also necessary for RCGP. Moreover, regression analysis yielded a 2-variable model that includes *logical reasoning and relevant geometry knowledge,* accounting for 54% of the variances on RCGP of 9th graders who had not learnt geometry proof in school as well as for 22% of the variances on RCGP of 10th graders who had learnt geometry proof in school. The difference between the two groups of students implied other variables are still required for students to understand proofs.

Language learning researchers considered that reading strategy use is an approach to improving reading comprehension. Moreover, many empirical studies showed that successful readers differ from less successful ones in both the quantity and quality of cognitive and metacognitive reading strategy use (Phakiti, 2003). Accordingly, Yang (2012) studied how students' perceived use of cognitive and metacognitive reading strategies is related to their RCGP. She found that cognitive reading strategies directly influenced students' RCGP and mediated between metacogntive reading strategies and students' RCGP. Good comprehenders tended to employ more cognitive reading strategies for elaborating proof and metacognitive reading strategies for planning and monitoring comprehension.

Moreover, motivation is one potential factor in examining students' utilization of those three dimensions in reading proofs. Hanna (1991) proposed that "the publication of a rigorous proof would provide no additional positive motivation for active acceptance, and in fact such a proof would not be examined at all in the absence of the motivating factors (p. 58)" from the view of mathematical practice. The motivation that students bring to reading proofs may be influenced by their proof conceptions (Healy & Hoyles, 2000). Future studies can investigate the relationship between students' attitudes towards doing and reading proofs in order to establish strategies for motivating students to read proofs from doing proofs and vice versa.

The context

Context is a pervasive and potent factor when considering any learning event (Tessmer & Richey, 1997). In the secondary literacy research, the context has been acknowledged as the learning environment for achieving different purposes of literacy development (Moje, Dillon & O'Brien, 2000). As purposes can influence which text is selected and how readers read, I particularly focus on purposes for mathematical reading to analyze the context. Two purposes are identified from past studies – for solving proof problems and for learning proofs.

Regarding the first purposes, studies related to solving word problems are most accessible. Given that understanding problems is the first step to successful problem solving (Polya, 1957), there is a high correlation between reading comprehension and mathematical problem solving (Aiken, 1972). However, similar studies are rarely found in solving proof problems. Cheng and Lin (2006) developed a reading-and-coloring strategy for helping students complete proof tasks. Students used this strategy to transform literal information into visual information, which further facilitated students' prior knowledge, and then to associate applied properties for proof. Studies on the effects of other scaffolding strategies on proof comprehension are necessary to promote rich mathematical communication.

One of the research questions related to the second purpose is how learning tasks can be designed for better understanding of mathematics proof. Yang and Lin (2012) considered that mathematics teachers were not familiar with the teaching of reading to learn mathematics, and thought of how to integrate reading into the learning of mathematical proofs. Hence, they designed reading-oriented tasks (ROT) based on reading strategies and the idea of problem posing. Results showed that the total scores of the delayed post-test of the experimental group using ROT were significantly higher than those of the control group. Furthermore, the scores of the experimental group on all facets of reading comprehension except the application facet were significantly higher than those of the control group for both the post-test and delayed post-test. Nonetheless, ROT did not actually improve students' RCGP because their performance in the post-test was a little worse than that in the post-test. Teaching of reading strategies may be worth to try as an approach to learning proofs.

Few studies are found to teach reading strategies to learn mathematics proof. Hodds, Alcock and Inglis (2014) explored the effect of self-explanation training on proof comprehension. A self-explanation is a form of self-talk where one generates explanations in an attempt to make sense of new information (Chi, 2000). They found that the self-explanation training was effective in understanding mathematics proof. Nonetheless, the effects of other reading strategies, such as predicting, questioning, summarizing and clarifying, on proof comprehension and other domains of mathematics are still under construction. For example, Su and Yang (2013) designed a teaching experiment to explore the application of the reciprocal teaching method in learning mathematics. A mathematics teacher and two classes of 90 eleventh graders who were taught by the same teacher participated in this quasi-experimental classroom study. The experimental group was instructed in line with the reciprocal teaching methods in six weeks, and each reading strategy was demonstrated by the teacher and practiced by students in the five sessions of one week. It was noted that the time on reading was no more than twenty-five minutes in each session. On the other hand, the control group was instructed with the regular method where the teacher explained and students exercised. The generalized estimated equation method was conducted on posttest and delayed posttest with pretest as a covariate. Results showed that the effect of the reciprocal teaching method on mathematics achievement and reading for learning mathematics decreased as time passed. The study implied that the value of the teaching of reading strategies in learning mathematics was still under construction.

Three perspectives on reading

According to the brief review on mathematical reading research related to proofs, different levels of students' performance on and pedagogical approaches to reading comprehension of proofs can be identified. Have those studies differentiated various meanings of mathematical reading? To answer this question, I firstly refer to the historical development of research on reading as to the views of behaviorists, Gestalt psychologists and social anthropologists (Alexander, & Fox, 2004; Borasi & Siegel, 2000). I would elaborate more on the three views.

For behaviorists, reading is to decode the marks on the page. To understand mathematical texts, reading requires both linguistic comprehension skills and knowledge of the 'language of mathematics,' which consists of prior mathematics knowledge and mathematics-specific reading skills (Mckenna & Robinson, 1990; Shard & Rothery, 1984). In this view, readers are viewed as always starting at the bottom, identifying signs, graphs or words, until they catch the text-bound meaning. Moreover, reading skills could be clearly defined and broken into constituent parts (Alexander, & Fox, 2004). For Gestalt psychologists, reading involves readers' interpretations of the text, depending upon their experiences and conceptual knowledge. In addition to obtaining information from the text, various meanings can be made as a result of the transaction between the reader and the text (Rosenblatt, 1978). Critical characteristic of this view involves making coherent sense of the text and an increased concern of the aesthetic stance toward the text. For social anthropologists, an additional impetus to evolve new view on reading is to shift the focus on reading from individualistic to social practice. It might be said that the learning outcome becomes less importance than the learning process (Sfard, 1988). The main goal of reading, like learning, is the creation of multiple meanings arising in the social interaction of particular individuals in a particular context. As Billings and Fitzgerald (2002) expressed, "the reciprocal flow of ideas involving actions and reactions of group members may lead to new understandings" (p. 909).

Miller (2007) has developed about three representative approaches to education: transmission, transaction, and transformation. The three approaches distinctively see the learners as the passive recipient of content, functional individuals in society, and collaborative participants for reconstructing society. Due to the similar ideas, three perspectives on reading are identified and termed as transmission, transaction and transformation. *Transmissive* perspective portrays reading as readers passively receive information of the text, *transactional* perspective focuses reading on meaning-making through the inter-subjective interaction between the reader and the text, and *transformational* perspective pays attention to the context which integrates the reader and the text to collaboratively participate in reading.

Multiple meanings of mathematical reading

Before proposing multiple meanings of mathematical reading, I classify mathematical texts into three types. The first type refers to the text that mainly consists of dense mathematical knowledge and is generally used for learning and teaching mathematics, e.g. mathematics textbooks or e-textbooks. The second type refers to the text where mathematical knowledge is partial and not the main focus, e.g. mathematical novels or science textbooks. The third type refers to the text that does not obviously present mathematics but can be read mathematically. By crossing the three types of mathematical text and the three perspectives on reading, nine categories of meanings of mathematical meaning are logically derived. However, it is impossible to generate meanings of the third type of mathematical text based on both transmissive and transactional perspectives as the two perspectives either wholly or partially rely on the text. Thus, seven categories of meanings of mathematical readings are produced. As to the first two types of mathematical text, the reader may either extract information from (transmissive perspective) or make meaning of (transactional perspective) either technically oriented mathematics or 'rich' mathematical texts (ref. Siegel, Borasi, & Smith, 1989). Herein, transmissive and transactional perspectives can respectively correspond to the views of reading as a set of skills in extracting information from the text and as a mode of learning (Borasi & Siegel, 2000). I further distinguish transformational perspective from transactional perspective in order to emphasize the potential of knowledge creation, the critical evolution of meanings in reading as a mode of learning. This perspective makes it possible to generate mathematical meanings from reading the text that does not obviously present mathematics.

Accordingly, mathematical reading can be viewed as the act of reading related to mathematics. The purposes of reading mathematical texts can be multiple and various.

Rather than advising against some perspectives, it is supposed that multiple meanings of mathematical reading are necessary for the learning and teaching of mathematics. After selecting a type of mathematical texts in some context, mathematical reading may be manifested to receive mathematical information, to make sense of mathematics and to mathematically transform the text into more and more meanings, no matter in individuals or in practices.

Suggestion

It can be concluded that: (1) multiple meanings of mathematics reading has not been realized in past work on reading mathematics proof; (2) there are few research studies on how students read mathematical texts in the practices of developing mathematical literacy although some studies have shown how mathematicians read mathematical proofs in practice; (3) both belief and affect factors are seldom investigated in reading mathematical proofs, and; (4) effective pedagogical approaches to learning proofs are under construction. Based on multiple meanings of mathematical reading, studies on mathematical reading in proofs can be extended to realize the practices of developing mathematical literacy. For instance, to put what is read in proof problems, students may need to draw graphs, integrate multiple information elements, transform formula logically, and explain the mathematical relationships between concepts and properties. Like studies on the relationship between reading and solving word problems, research can investigate students' proof behaviors from multiple perspectives on mathematical reading to find the characteristics of recognizing the given information (transmission), rewriting it on paper as a new text (transaction) or turning a conjecture into lemmas or a theorem by proving (transformation).

Mathematics education researchers have aimed at conceptualizing multiple meanings of reading comprehension of mathematical texts (e.g. Duru & Koklu, 2011; Mejia-Ramos et al., 2012; Yang & Lin, 2008) or modifying the text for students to learn proofs (e.g., Chen & Yang, 2010; Yang, Lin and Wang, 2008). Those studies viewed reading comprehension as part of mathematical understanding; however, the relationship between reading comprehension and mathematical understanding can be intertwined and dual. For example, Borasi, Siegel, Fonzi and Smith (1998) implemented transactional reading strategies to lead secondary school students to engagement in mathematical inquiry. Pape (2004) synthesized mathematical problem solving and reading comprehension theories to investigate children's problem-solving behaviors. Those studies integrated reading with mathematics learning.

In sum, mathematics education researchers operating from any one category of mathematical reading can aim at developing mathematical literacy involving constructing, applying and evaluating knowledge in the practices of learning by reading. Such multiple processes enhance students' power to draw on their knowledge and to raise meaningful critiques of what they learn. The more students engage in the practices of learning mathematics by reading, the more they can become flexible and constructive readers among various types of mathematical texts. For reaching the purpose, new considerations in future research are the need to develop multiple meanings of mathematical reading, the role of belief and affect in mathematical reading, and the need for more teaching experiments and more programs in teacher professional development based on multiple meanings of mathematical reading.

References

Alexander, P. A., & Fox, E. (2004). A historical perspective on reading research and practice. *Theoretical models and processes of reading*, *5*, 33-68.

Aiken, L. R. (1972). Language factors in learning mathematics. *Review of Educational Research*, *42*(3), 359-385.

Bernardo, A. (1999). Overcoming obstacles to understanding and solving word problems in mathematics. *Educational Psychology*, *19*, 149-163.

Billings, L., & Fitzgerald, J. (2002). Dialogical discussion and the Paideia seminar. *American Educational Research Journal*, *39*, 907-941.

Bloome, D., & Egan-Robertson, A. (1993). The social construction of intertextuality in classroom reading and writing lessons. *Reading Research Quarterly*, *28*(4), 305-333.

Borasi, R., and Siegel, M. (2000). *Reading counts: Expanding the role of reading in mathematics classrooms*. New York: Teachers College Press.

Borasi, R., Siegel, M., Fonzi, J., & Smith, C. F. (1998). Using transactional reading strategies to support sense-making and discussion in mathematics classrooms: An exploratory study. *Journal for Research in Mathematics Education*, 275-305.

Chen, Y. H, & Yang, K. L. (2010). The effects of self-reading and text layouts on seventh graders' mathematical performance about "Pythagorean Theorem". *Journal of Research in Education Sciences*, *55*(2), 141-166.

Cheng, Y. H., & Lin, F. L. (2006, July). Using Reading and coloring to enhance incomplete prover's performance in geometry proof. In *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 289-296).

Conradie, J., & Frith, J. (2000). Comprehension tests in mathematic. *Educational Studies in Mathematics*, *42*(3), 225-235.

Duru, A. & Koklu, O. (2011). Middle school students' reading comprehension of mathematical texts and algebraic equations. *International Journal of Mathematical Education in Science and Technology*, *42*(4), 447-468.

Hanna, G. (1991). Mathematical proof. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 54-61). Norwell, MA: Kluwer.

Healy, L. & Hoyles, C. (2000). From explaining to proving: a study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*, 396-428.

Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education*, *45*(1), 62-101.

Lerkkanen, M.-K., Rasku-Puttonen, H., Aunola, K., & Nurmi, J.-E. (2005). Mathematical performance predicts progress in reading comprehension among 7-year olds. *European Journal of the Psychology of Education*, *20*(2), 121-137.

Lin, F. L. & Yang, K. L. (2007). The reading comprehension of geometric proofs: The contribution of knowledge and reasoning. *International Journal of Science and Mathematics Education, 5*(4), 729-754.

McKenna, M. C., & Robinson, R. D. (1990). Content literacy: A definition and implications. *Journal of Reading*, *34*(3), 184-186.

Mejía-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, *79*, 3-18.

Miller, J. P. (2007). The holistic curriculum (2nd ed.). Toronto, Canada: University of Toronto Press.

Moje, E. B., Dillon, D. R., & O'Brien, D. (2000). Reexamining the roles of learner, text, and context in secondary literacy. *Journal of Education Research*, *93*, 165-180.

Morgan, C. (1996). *Writing mathematically: The discourse of investigation*. London: Falmer Press.

Pape, S.J. (2004). Middle school children's problem-solving behavior: a cognitive analysis from a reading comprehension perspective. *Journal for Research in Mathematics Education*, *35*(3), 187-219.

Piaget, J. (1951). *Origins of intelligence in children*. London: International University Press.

Polya, G. (1957). *How to solve it: A new aspect of mathematical method*. Garden City, NY: Doubleday Anchor Books.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning*, *5*(1), 37-70.

Rosenblatt, L. (1978). *The reader, the text, the poem: The transactional theory of literary work*. Carbondale, IL: Southern Illinois University Press.

Rosenblatt, L. M. (1986). The aesthetic transaction. *Journal of Aesthetic Education*, 122-128.

Rumelthart, D. E. (1994). Toward an interactive model of reading. In R. B. Ruddel, M. R. Ruddell, & H. Singer (Eds.) *Theoretical models and processes of reading (4th ed.)*. Newark, DE: International Reading Association.

Schleppegrell, M. J. (2010). Language in mathematics teaching and learning. A research review. In J. Moschkovich (Ed.), *Language and mathematics education* (pp. 73–112). Charlotte: Information Age Publishing.

Siegel, M., Borasi, R., & Smith, C. (1989). A critical review of reading in mathematics instruction: The need for a new synthesis. In S. McCormick & J. Zutell (Eds.), *Cognitive and social perspectives for literacy research and instruction: The 38th yearbook of the National Reading Conference* (pp. 269-277). Chicago: National Reading Conference.

Shuard, H., & Rothery, A. (Eds.). (1984). *Children reading mathematics.* London: John Murray.

Tessmer, M., & Richey, R. C. (1997). The role of context in learning and instructional design. *Educational technology research and development*, *45*(2), 85-115.

Tierney, R. J., & Pearson, P. D. (1992). A revisionist perspective on learning to learn from text: A framework for improving classroom practice. In R. B. Ruddell, M. R. Ruddell, & H. Singer (Eds.), *Theoretical models and processes of reading* (pp. 514-519). Newark, DE: International Reading Association.

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, *30*(2), 161-177.

Vygotsky, L.S. (1962). *Thought and Language*. Cambridge, MA: MIT Press.

Yang, K. L. (2012). Structures of cognitive and metacognitive reading strategy use for reading comprehension of geometry proof. *Educational Studies in Mathematics*, *80*(3), 301-326.

Yang, K. L., & Lin, F. L. (2008). A model of reading comprehension of geometry proof. *Educational Studies in Mathematics*, *67*(1), 59-76.

Yang, K. L., & Lin, F. L. (2012). Effects of reading-oriented tasks on students' reading comprehension of geometry proof. *Mathematics Education Research Journal*, *24*(2), 215-238.

Yang, K. L., Lin, F. L, & Wang, Y. T. (2008). The effects of proof features and question probing on understanding geometry proof. *Contemporary Educational Research Quarterly*, *16*(2), 77-100.

Acknowledgement
The author wishes to thank all of the participating researchers, teachers and students for their efforts in research on mathematical reading. This review paper is based on a collection of research projects funded by the National Science Council of Taiwan.

Kai-Lin Yang
National Taiwan Normal University
Department of Mathematics, No. 88, Ting-Chou Rd. Sec. 4, Taipei, Taiwan
kailin@ntnu.edu.tw