Conjecturing and proving as an approach to facilitate active thinking and peer discussion in combinatorial class

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Introduction and aims

Teaching and learning in combinatorics

Although combinatorics plays an important role in probability learning, for many students and teachers, combinatorics is nothing more than mathematical symbols and enumerative techniques. That's why many kinds of learning difficulties and students' mistakes are reported in literature (Batanero, Navarro-Pelayo, & Godino, 1997). To improve the aforementioned situation, we should take both the essence of combinatorial problems and students' difficulties into consideration at the same time.

In Soviet tradition, enumeration was classified into two categories: "constructive enumeration" and "analytic enumeration." The former means the creation of a complete list of all objects of a desired type. The latter, in contrast, is simply determining the cardinality of such a list (Ebert, Ebert, & Klin, 2004).

In Taiwan, students in elementary schools were encouraged to use constructive enumeration to solve problems. Nevertheless, because of its inefficiency, constructive enumeration was not encouraged or even prohibited in high schools. Most students' errors in combinatorial problem solving resulted from their lack of number in enumeration and misuse of combinatorial formulas. We thus conjectured that connecting "constructive enumeration" and "analytic enumeration" might improve students' number sense in enumeration.

From acquisition to participation in mathematics learning

Sfard (1998) pointed out the importance of balance between acquisition and participation in mathematics learning. In tradition, Taiwanese mathematics teachers usually introduced concepts and demonstrated procedures while students listened, took notes, and practiced in class. Students acquired knowledge from Sfard's acquisition metaphor. Owing to students' lack of confidence in mathematics learning revealed in TIMSS 2003, Lin (2006) stressed the importance of active learning. Upholding the belief that teachers should think actively before students do, he proposed a framework for designing conjecturing (FDC). In FDC, there were three entries for designing conjecturing activities: false statement, true statement, and learners' conjecture.

Lin, Yang, Lee, Tabach & Stylianides (2012) conducted further research and then proposed four principles for conjecturing: observation, construction, transformation, and reflection. Within conjecturing activities, students were offered opportunities to observe particular cases, construct new knowledge, transform prior knowledge, and reflect on their conjecturing processes.

Research aims and questions

The aims of the study were twofold. First, we wanted to explore the effects of the conjecturing activities in a combinatorial class. Second, followed by the framework (Lin, 2006) and the principles (Lin et al., 2012) for conjecturing task designing, we would like to explore teachers' intervention both in conjecturing task designing and in class. Two research questions were proposed in this study: What kind of students' active thinking behaviors could be observed in the combinatorial class guided by the conjecturing principles? What kind of teachers' intervention may foster students' jump in their zone of proximal development (ZPD)?

Literature review and theoretical perspective

Teaching strategies for combinatorial problem

From the problem-designing perspective, task variables were selected as didactic variables in combinatorial teaching. For example, Batanero, Navarro-Pelayo, & Godino (1997) used an implicit combinatorial model as their theoretical framework and concluded that the combinatorial model should be considered as a didactic variable in organizing elementary combinatorial teaching.

From the pedagogical perspective, English (2005) found that children always focused on context features instead of mathematical structure of problems. Sriraman & English (2004) also suggested "choosing problems that vary contextually but are essentially similar in their mathematical structure is pedagogically important."

From the problem-solving perspective, Eizenberg & Zaslavsky (2004) reported students' verification strategies for combinatorial problems and found that one effective strategy was applying "the same solution method by using smaller numbers." Lockwood (2013a) found the method facilitated systematic listing of outcomes and resulted in meaningful counting processes, which led to appropriate formulas for the situation.

From the perspective of social constructivism, Eizenberg & Zaslavsky (2003) used collaborative learning as a teaching strategy and found the interrelationship between collaboration, control, and successful problem solving.

In conclusion, most research in combinatorial teaching only focused on one phase of mathematics learning activity: problem solving. While the foundation of combinatorial thinking, the strategies for teaching combinatorial concepts, has yet to be emphasized in the empirical research of combinatorics.

The ZPD framework

Goos (2004) led students to conduct inquiring-based learning in an advanced math class in Australia for two years and meanwhile collected data to represent students' ZPD. Later, she proposed a ZPD framework for analyzing students' learning, she also explored three functions of ZPD: as scaffolding for teacher-student interaction, collaboration for student-student interaction, and interweaving for everyday and scientific concepts. Since the framework was suitable to describe students' participation in the conjecturing class we conducted, we adopted it as our theoretical framework to analyze students' active thinking behaviors.

The model for combinatorial thinking
In order to understand how students' conceptualized counting problems, Lockwood (2013b) proposed a model for combinatorial thinking from literature review and his own interview data. The model contained three components: the counting processes, the sets of outcomes, and the expressions/formulas; detailed descriptions were as follows:

Counting processes: refer to the enumeration process in which a counter engages as they solve a counting problem.

Sets of outcomes: refer to the collection of objects being counted —those sets of elements that one can imagine being generated or enumerated by a counting process.

Expressions/Formulas: refer to mathematical expressions that yield some numeral value. Two expressions may be mathematically equivalent but differ in forms.

Lockwood (2013b) further suggested researchers using the model to describe and analyze students' thinking in counting activities. Nevertheless, the relationship between sets of outcomes and expressions/formulas was still under investigation.

Therefore, one of our research aims was to analyze students' counting activity from specialization to generalization, which might contribute to the connection between sets of outcomes and expressions/formulas.

Research designs and methods

Participants and setting
Thirty-eight 10th grade students in a public school in Taiwan (PR=70 ~80) participated in the combinatorial class taught by the second author. The second author is a senior high school teacher with 12 years of teaching experience. Because of the student-centered belief, he has designed and conducted conjecturing activities in the same class for one semester. He initiatively invited the first author, some schoolteachers, and also a mathematical educator to participate in the classroom observation.

All students were grouped and did the conjecturing tasks designed by the second author, each group with 4 students. Those schoolteachers, who served as participant observers, also divided into groups.

Data collection
Classroom observation and video recordings of teacher-student & student-student interactions were conducted in the classroom. At the same time, each participant observer took field notes to record the teacher-student & student-student interactions. After class, the mathematics educator directed the discussions among observers.

Data analysis
To ensure the validity of findings, all the collected data were validated by triangulation among the first author, the second author, and the educator.

Conjecturing Tasks
The worksheet "Methods for Choosing People" was presented and discussed in class as follows:

1. How many ways can you choose 2 people from a 3-people group?
   (Please list all the results.)
2. How many ways can you choose 2 people from a 4-people group?

(Please list all the results.)
3. How many ways can you choose 2 people from a 6-people group?
(Please list all the results.)
4. How many ways can you choose 2 people from an n-people group?
(Generate a conjecture and try to elaborate it.)

Findings and discussions

From specialization to generalization

The first two problems were easy for students to count and the teacher encouraged students to "list all the results," students were able to finish the tasks on their own. Even though the last two problems were relatively hard, students could also finish the task through mutual collaboration.

Specialization

The students solved the first two problems by systematic counting or over-counting which as follows:

Method 1 (systematic counting):

AB, AC, AD
BC, BD
CD
=> 6 methods.

Method 2 (over-counting and dividing):

AB, AC, AD
BA, BC, BD
CA, CB, CD
DA, DB, DC

=> $\frac{4\times 3}{2}$ = 6 methods.

Generalization

With the experiences in problems 1, 2, and 3 most students conjectured that the answer for problem 4 was $\frac{n(n-1)}{2}$ or $\frac{n^2-n}{2}$. Noticing the students all made their conjectures, the teacher encouraged three students to write down their conjectures on the board and elaborate them in public.

S1: Since there are n people and every person has (n-1) choices (where AB and BA are the same), so totally we have $\frac{n(n-1)}{2}$ kinds of choices (Figure 1).

The logic of S1 was totally correct though his explanation to generalization might be too fast so that some students could not catch up with him. Thus, the teacher encouraged the other students to give supplementary explanations.

S2: Take problem 3 as an example, now we have 6 people. Since A can be coupled with 5 people: B,…., and F. While B can be coupled with 4 people: C,.., and F. And C can be coupled with 3 people, … and so on; we thus have 5+4+3+2+1= 15 choices. Hence, if we have n people: the total result goes to $(n-1)+(n-2)+...+1 = \frac{(1+n-1)\times(n-1)}{2}$ (Figure 2).

Figure 1. Enumerating by the meaning of the context

Figure 2. Enumerating by the trazepoid formula

S2 used the special case with 6 people to explain, which led to the general formula of n people. The explanation was clearer. By comparing S1's and S2's explanation, the teacher asked students to go even further.

S3: We can use a table to connect the previous two explanations. Since there are n people, and they can choose every person as their couple except themselves, we thus have ($n\times n-n$) choices. Since the outcome is symmetric, which means AB and BA are the same. As a result, we have $\frac{n\times n-n}{2}=\frac{n^2-n}{2}$ kinds of choices (Figure 3).

Figure 3. Enumerating in the table representation

The teacher's intervention

The teacher's intervention could be analyzed by the four principles for conjecturing (Lin et. al., 2012). First, students were asked to list all the sets of outcomes and observe the results in the special cases. Second, students were offered opportunities to construct their own conjecture of the counting formula in the general case. Third, students were offered opportunities to transform between different formulas. Finally, by means of transformation, students' were offered opportunities to reflect on the meanings of the enumerative formulas.

Evidence for students' cognitive jump in their ZPD
Based on Lockwood' model for students' combinatorial thinking (2013b), we could analyze students' counting activity as follows:

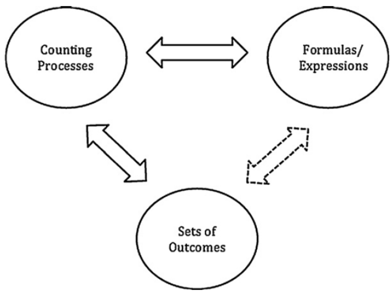

Figure 4. A model of students' combinatorial thinking (Lockwood, 2013b)

From counting processes to sets of outcomes: ZPD as an interweaving
The teacher proposed the counting problems and asked the students to list the sets of outcomes. The teacher insisted on "listing all the outcomes" to ensure students listed all outcomes by one feature of the counting situation. Due to different counting processes, the students listed two different expressions: the trapezoid form or the table form. The finding supported ZPD as interweaving between daily and scientific concepts.

From sets of outcomes to formulas/expressions: ZPD as scaffolding
In the counting activity documented by the worksheet and mentored by the teacher, the students experienced a discovery process from specialization to generalization. From sets of outcomes, the students made a progress in the expressions of special cases and even a formula of the general case. Their formulas were in two forms: by the trazepoid formula $\frac{(1+n-1)\times(n-1)}{2}$ or by a table listing $\frac{n\times n-n}{2}=\frac{n^2-n}{2}$. The finding supported ZPD as scaffolding for teacher-student interaction.

Relationship between different formulas/expressions: ZPD as collaboration
When the students were asked to write down various results on the board and elaborated in public; they had the chances to help each other. That was because some students' explanations were not easy to be understood, and the others were willing to make them clearer. This showed when different formulas/expressions occurred, students would have the chance to compare various methods and try to connect them. The finding supported ZPD as collaboration for student-student interaction.

Conclusions
FDC supported teachers' intervention in conjecturing activity
Theoretically, the second author designed the tasks based on FDC. Empirically, we found the tasks offered students opportunities to observe (features of the sets of outcomes), to construct (their own counting formulas for the situation), to transform (from counting processes to formulas), and to reflect (through interpretations of

different formulas). As a result, we could say that FDC offered students opportunities to communicate, collaborate, and discuss within the conjecturing activities. To sum up, FDC functioned as an intermediate framework for the conjecturing-based learning setting.

FDC offered students' opportunities to jump in their ZPD

In the class, the second author asked students to write down their counting formulas on the board and elaborate. We found different students might apply different formulas. With the spirit of FDC, the second author asked the students to compare their results with each other, which offered students opportunities to broaden their views from transforming different representations and reflecting on their own counting experiences. In conclusion, FDC offered students' opportunities to jump in their ZPD.

FDC supported students' combinatorial thinking

The counting problems were designed with the spirit of FDC, which offered students opportunities to observe special cases and transform them to a general case. With the aid of table representations or learned formula, evidenced were found to prove that students actively transformed their sets of outcomes in special cases to formulas in a general case. To sum up, FDC supported students' combinatorial thinking.

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