Research on development of the prescriptive model for designing social interactions in elementary school mathematics classrooms

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1. Introduction

The development of the ability to think, judge and express oneself has been emphasized in the current Japanese curriculum revised in 2008. In line with the curriculum policy behind this revision, many mathematics education researches and mathematics teaching practices have emphasized such children's activities as thinking mathematically by using various representations, and explaining and communicating their mathematical ideas among them. However, it seems that most of previous researches in question are mere case studies, whose theoretical background is not clear and perhaps limited; and the roles teachers must play when helping children communicate with each other are not described well, because those researches and teaching practices often take a child-centered approach and do not consider the teacher sufficiently.

To overcome these issues, this research aims to develop the prescriptive model for designing *social interactions* in elementary mathematics classrooms that can be effective in and applicable to teaching practices at this level. We verify the effectiveness of this model through a teaching experiment on the topic of fractions, conducted by one teacher in two fourth-grade classrooms at the same elementary school in Japan.

2. Theoretical framework of this research: The prescriptive model for designing social interactions in elementary mathematics classrooms

Our theoretical framework, which presents a "prescriptive model for designing social interactions in elementary mathematics classrooms," (hereafter called *"the prescriptive model"*) is shown as Fig.1. The main characteristics of our model can be summarized as follows:

1) The epistemological background is the so-called *multi-world paradigm* of mathematics education.
2) The prescriptive model has a *fundamental process* consisting of five key stages; *being conscious*, *solving in small group*, *solving individually*, *being reflective,* and *forming agreement*. Small group activities are emphasized to promote social interactions.
3) Three kinds of social interactions were assigned to each stage of the fundamental process: social interaction with *others* (social interaction A), social interaction with *oneself* (social interaction B) and social interaction with *representations* (social interaction C).

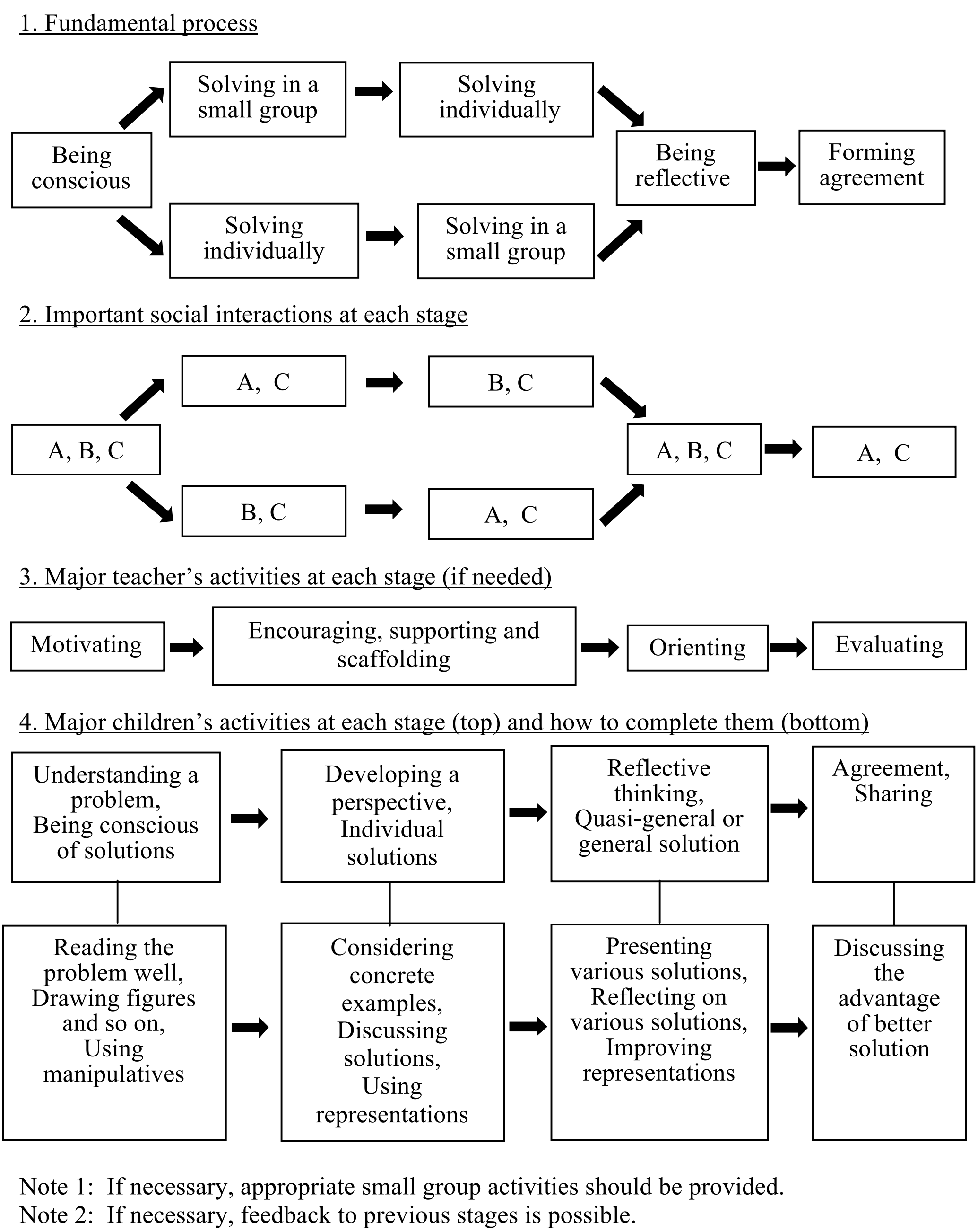


Figure 1. Prescriptive model for designing social interactions in elementary mathematics classrooms

4) Major children’s and teacher’s activities are allocated to each stage.
5) Three levels of generality of children’s solutions are considered at *being reflective*: *individual solution*, *quasi-general solution* and *general solution*.

We first give some supplementary explanation of Figure 1.

Characteristic 1) shows the epistemological background of the prescriptive model. Nakahara (1999) proposed the so-called *multi-world paradigm* of mathematics education that coordinates three epistemologies: *radical constructivism*, *interactionism* and *socioculturism*. Because the multi-world paradigm helps us accurately capture realities of children's learning of mathematics, the prescriptive model also adopts it as an epistemological foundation.

Regarding characteristic 2), the teaching and learning process is an integral part of the prescriptive model. From this perspective, we adopted the *constructive approach* that Nakahara (1995) proposes; slightly modified it, adding small group discussions; and established it as the fundamental process of the prescriptive model as shown in Fig.1. The children’s main activities are as follows.

a) *Being conscious*: Children are conscious of the problem they need to solve.
b) *Solving in a small group* ≥ *Solving individually*: Children discuss ways of solving the problem in a small group. Then, they solve the problem individually.
c) *Solving individually* ≥ *Solving in a small group*: Children solve the problem individually. Then, they present their solutions and discuss them in small groups.
d) *Being reflective*: Children present their own ways of solving the problem. After the presentation, they discuss and refine their approaches with the whole class.
e) *Making agreement*: Children agree which ways of solving the problem are best and summarize them.

The term “social interaction” mentioned in characteristic 3) is mainly used to mean interactions among children, or between children and a teacher. However, we intentionally distinguish social interactions, A, B, and C, as defined above, and do not limit these interactions to interactions with others because of the importance of social interactions in mathematics lessons. Social interaction with oneself is a kind of self-dialogue. For example, it means that a child reflects on his/her ideas and recognizes cognitive conflicts for oneself. Social interaction with representations means that any change in children’s ideas and exchange of them among children are promoted when they encounter various representations. In Fig.1, we propose positioning these social interactions as shown to promote the children's learning activities at each stage.

Regarding characteristic 4), we think that a teacher should play appropriate roles in mathematics lessons. From this viewpoint, we assigned teacher’s roles at each stage in Fig.1.

Regarding characteristic 5), to support children’s activities at *being reflective*, we set up three levels of generality of children’s solutions, as follows.

(Level 1) *Individual solution*: Solution of the original problem
(Level 2) *Quasi-general solution*: A more generalized solution using concrete numbers or examples
(Level 3) *General solution*: A fully general solutions using letters and words

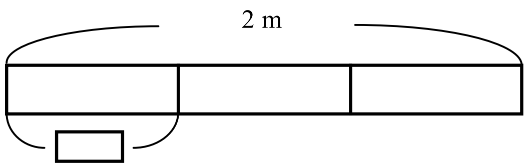
3. Teaching experiment on "fractions" in fourth-grade classrooms

We designed several teaching experiments to verify our prescriptive model. This experiment for fourth-graders, on the topic of "fractions," is one of them.

Outline of the teaching experiment

The following fraction problem was considered in the teaching experiment.

There is a tape whose length is 2m. This tape is divided into three equal parts, as in the following figure. How can we represent $\square$?



Although the correct answer is "2/3m," many children often make a mistake and answer "1/3m" because the tape is divided into three equal parts. The relation between fractions children must understand to solve this problem is that 1/3 of a tape whose length is 2m is 2/3m; thus, they must differentiate two kinds of meanings of fractions (a fraction of a given whole, that is, a fraction of 1, and a fraction of a defined quantity other than 1). However, understanding this relation is difficult for elementary school children. To take this difficulty into account, we planned our teaching experiment.

The teaching experiment was conducted in three fourth-grade classrooms in a public elementary school in the city of Kagoshima in Japan, from November 26 to December 6, 2013. It required two lessons (each 45 minutes long) in each classroom. An expert teacher taught all three classrooms (all six lessons). In this paper, we focus on two of three classrooms, hereafter called, "Class A" and "Class B" because the children's activities in Class A make a striking contrast to those in Class B.

Hypotheses of the teaching experiment and criteria for verification of its effectiveness

We formulated four hypotheses to verify the effectiveness of the prescriptive model, as shown in Table 1. How these hypotheses contribute to the activation of social interactions among children and how social interactions contribute to deepening children's understanding of fractions are important points for the verification of our model. From this standpoint, we set up four criteria to evaluate children's understanding of fractions (Table 2).

Table 1. Four hypotheses

a) Hypotheses for the first lesson

Hypothesis 1	If a teacher gives children this open-ended problem at *being conscious*, they will be able to propose various ideas of representing $\boxed{}$ (hereafter, “the blank”), such as 1/3, 1/3m and 2/3m.
Hypothesis 2	If a teacher encourages children to review the definition of fractions already learned in the third grade, children will have doubts about the “1/3m solution” and will gradually realize a cognitive conflict against it. Then, they may propose the idea of 2/3m and consider why 2/3m is the correct answer.

b) Hypotheses for the second lesson

Hypothesis 3	If a teacher gives children an illustrative representation of a tape whose length is 1m (hereafter, the “1-m tape”), when they discuss which of 1/3m and 2/3m is the correct answer, they will gradually come to have more confidence in the 2/3m option and to be able to clearly explain why it is correct. Then, they will be able to share their solutions and agree that the best solution is the one that reflects the relation between concepts of fractions: that the length of 1/3 of one tape whose length is 2m is 2/3m.
Hypothesis 4	If a teacher gives applied problems with variables such as the length of a tape and the number of parts into which is divided to children who can clearly explain why 2/3m is the correct answer to the problem given above, they will be able to generalize solutions to the original problem and find ways of solving them.

Table 2. Four criteria for the evaluation of the understanding of fractions

Criterion 1	Whether children realize a cognitive conflict against the answer of 1/3m.
Criterion 2	Whether children can propose the idea of 2/3m and also explain why it is correct.
Criterion 3	Whether children can understand the relation between concepts of fractions and differentiate between meanings such that they understand that solution to the problem is that the length of 1/3 of one tape whose length is 2m is 2/3m.
Criterion 4	Whether children can generalize solutions to an original problem by focusing on the length of a tape or the number of parts into which it is divided.

4. Brief report of lessons in Class A and Class B

We’d like to give a brief report of two lessons here by comparing Class A with Class B, focusing on similarities and differences between them.

The first lesson

It was confirmed that the outline of the first lesson was almost the same in Class A and Class B. Concretely, at *being conscious*, the teacher gave the problem to his children and set up the objective of the lesson with them: “How can we represent the blank?” Then he encouraged children to solve it individually. After this, some children proposed the idea of 1/3 and 1/3m in the whole class discussions on the basis of their individual efforts; however, other children took a skeptical view of this answer. The teacher, therefore, organized small groups and asked the children to discuss these solutions. Then he wrote "1/3: One of three equal parts which one tape is divided into" on the blackboard.

At *being reflective*, after small group discussions, some children realized a cognitive conflict against 1/3m and gradually became confident in the answer of 2/3m. However, they could not explain why 2/3m was correct. So, at *forming agreement*, the teacher and his children summarized what they had learned in the following two points: a) There were two kinds of ideas advanced, relating to 1/3m and 2/3m. b) The class couldn't clearly explain why 2/3m was correct, although doubts around 1/3m increased.

The second lesson
At *being conscious* in the second lesson, the teacher first reminded his children of the event of the previous lesson, in both Class A and Class B. Then he set up the objective of the second lesson, that is, "We will try to find the correct answer and explain the reason it is correct." After this introduction, the teacher asked the children to discuss the correct answer and try to explain why it was correct in small groups.

As a result of small group discussions in Class A, all groups adopted the 2/3m solution. Furthermore, one boy skillfully explained the reason why 2/3m was correct using two kinds of illustrative representations of two 1-m tapes and one 2-m tape. Most of the children clapped their hands in applause and agreed with his explanation. Inspired by this child's idea, another boy presented an additional explanation with numerical expressions, as follows: "Three 1/3m's makes 1m. We have one more 1-m tape now. So we have six 1/3m's. We can get two 1/3m's when a 2-m tape is divided into three parts. So the answer is 2/3m." Then the teacher and all children negotiated a consensus answer: the length of 1/3 of one tape whose length is 2m is 2/3m. Furthermore, at the end of the second lesson in Class A, the teacher gave his children applied problems in which the number of parts into which a 2-m tape was divided was four and one hundred, respectively. The children solved them and explained reasons easily.

In contrast, in Class B, many children had asserted that the answer was 1/3m from the beginning to the end, although some children had doubts here as well. Therefore, the objective of the lesson was not attained well in Class B.

5. Analysis of lessons and some implications for the improvement
The results of our teaching experiment with fourth-graders are summarized in Table 3. We want to note here that the failure to confirm the hypothesis 4 in Class B was primarily caused by a failure to confirm the hypothesis 2 and 3. Thus, we thought, a further discussion of improvement of this activity for teaching fractions should focus on hypotheses 2 and 3.

For the improvement of the teaching of fractions, we can point out two suggestions that may be derived from the comparative analysis of two classes. The first, which is closely related to the hypothesis 2, is the importance of teaching the *"additive view"* of the definition of fractions. In our teaching experiment, the teacher reviewed the definition of fractions as follows: "There is a 1-m tape that is divided into four equal parts. One of them is written as 1/4m.” This definition is based on *the activity of dividing a unit quantity into some number of equal parts*. We refer to this as the fundamental feature of the “*dividing view”* of fractions.

Table 3. Results of the verification of four hypotheses in terms of four criteria

Hypotheses	Class A	Class B	Overall judgment
1. Effect of presenting open-ended problem at *being conscious*	○	○	○
2. Effect of reviewing the definition of fractions	○	—	△
3. Effect of two kinds of illustrative representation	○	—	△
4. Effect of presenting applied problems	○	not tested	△

(Note) ○: positive effect, —: little effect, △: partial but insufficient effect

The “additive view,” in contrast, means that in the case of 1/4m, adding four 1/4m’s makes 1m (1/4m+1/4m+1/4m+1/4m=1m). It seems to us that the additive view is reversible and almost the same as the canonical definition of fractions. We found that the children in Class A could demonstrate the additive view well but that the children in Class B could not. This seems to indicate that teachers introducing fractions should emphasize the additive view as well as the dividing view.

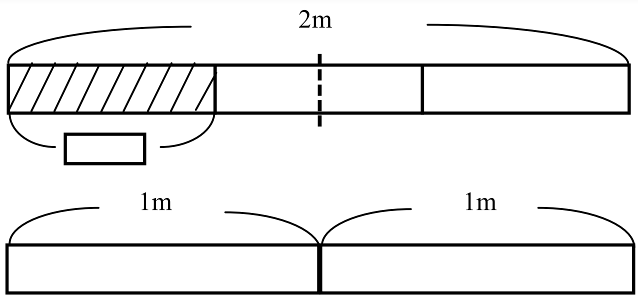
The second suggestion, which is closely related to the hypothesis 3, is of the importance of realizing two kinds of illustrative representations namely, two 1-m tapes and one 2-m tape. Children need to take two steps using these representations to solve the problem. First, they must realize that two 1-m tapes are embedded in one 2-m tape, which will lead them to find two abstract 1-m in one 2-m length (Fig. 2(a)). The second step is to understand that there are three 1/3m’s in each 1-m, using the additive view, and that the length of 1/3 of a 2-m corresponds to two 1/3m’s (Fig. 2(b)). It was confirmed that many children in Class A could understand and explain these structural facts effectively by using and combining tapes. However, the children in Class B only put *one* 1-m tape and one 2-m tape side by side, and could not find *two* 1-m in one 2-m. Considering these differences between two classes, teachers should provide children with support to visualize the fact that two 1-m tapes are embedded in one 2-m tape, by marking the 1m point on the tape with some sign, as shown in Fig. 2(a).

6. Conclusion

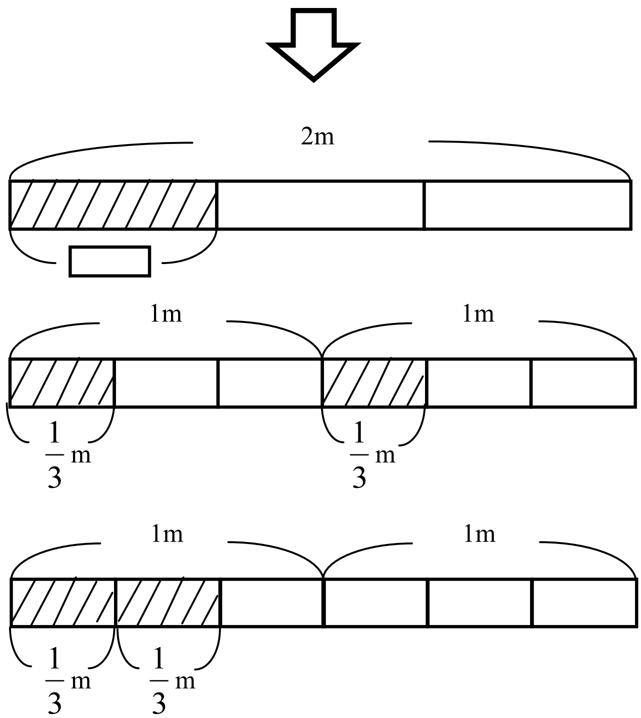
In this paper we verified the effectiveness of the prescriptive model through a teaching experiment. Qualitative analysis yielded the following four results.

Firstly, the fundamental process in Fig.1, in particular the small group activity, contributed to solving the problem at hand. Also, it was suggested that the children’s solution progressed from their “individual” solution to the “quasi-general” one.

Secondly, the teacher’s efforts to provide support using illustrative representations were quite effective. Some children in one classroom could solve the problem of fractions for themselves, and explain clearly why their answer was correct by using two kinds of tapes. In addition, the children negotiated the solution that the length of 1/3 of one tape whose length was 2m was 2/3m; thus, they differentiated two kinds of meanings of fractions. Furthermore, the children translated illustrative representations into symbolic representations, and they could generalize their solution to other applied problems.



(a) The first step of recognition



(b) The second step of recognition

Figure 2. Recognition of the structure of two 1-m tapes with one 2-m tape

Thirdly, three kinds of social interactions were activated by organizing the children in small groups. These various kinds of social interactions helped deepen the children's understanding of fractions.

Lastly, we made two concrete suggestions for the improvement of teaching fractions on the basis of this comparative analysis of two classrooms. First, it is crucial for the children to develop the additive view of the definition of fractions to solve the problem. Second, it is important for the children to realize two steps of the structure, which is embedded in illustrative representations of tapes.

Although these four findings demonstrate a certain degree of effectiveness of the prescriptive model, they are limited to the teaching experiment on the topic of fractions. Therefore, it is the future issue that we will continue to verify the effectiveness of the prescriptive model through some experiments on the other topics of teaching materials.

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