Affordances and constraints by mathematical tasks, questions, prompts and teacher actions: Insights from a lesson on fractions
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Background
Given the complexity of the interaction between the teaching and learning processes in a mathematics lesson, much research attention has been geared towards understanding in situ what mathematics teachers do as they teach and interact with students. The affordances and constraints initiated by the teacher can be a powerful tool to assess the extent to which mathematical concepts are made accessible to students (Watson, 2007). Similarly, the unpredictable constraints generated in a lesson can shed much light in terms of the impediment to learning emanating from teaching actions.

This case study reports the mathematical affordances initiated by an experienced teacher as he developed an introductory lesson on fractions in a Grade 7 class. Our moment-by-moment analysis of the lesson trajectories traces the constraints generated by the experienced teacher as he attempted to prompt students to develop the concept of fractions. We address the following research question—an issue more generally raised by Watson (2007): What opportunities to act mathematically are afforded and constrained by tasks, questions, prompts and teacher actions in the teaching of fractions?

Research on teaching fractions: A brief overview
Mathematics educators have found that rational numbers are not only problematic for students but also for teachers as far as its instruction is concerned (Post, Harel, Behr, & Lesh, 1991). Hackenberg’s (2007) explanation about the challenge for students of constructing improper fractions provides further support this finding. It requires students to extend or modify their part-whole understanding of fractions to interpret it as a number in its own right, for instance, interpreting 12/5 as both a mixed number and a number in its own right.

Teachers’ manoeuvres in fraction instruction are an issue that continues to attract research attention (e.g., Izsak, Jacobson, de Araujo, & Orrill, 2012; Rianasari, Budayasa, & Patahuddin, 2012). While there has been a separate focus on students’ and teachers’ (especially pre-service teachers’) knowledge of fractions, some researchers have specifically studied the interaction between the teaching and learning of fractions by analysing how teachers respond to students’ questions (Izsak et al., 2012). In line with efforts to understand how teachers use mathematical tasks, initiate questions, generate prompts and act in response to students’ responses, this study scrutinizes the teaching of fractions using Watson’s framework as an analytical tool.

Watson’s analytical framework
In dissecting the concurrent processes of mathematics teaching and learning, a number of constructs have been proposed to capture the multifaceted dimensions of classroom
events, such as the role of tasks, revoicing, interrogating meaning and noticing (Franke, Kazemi, & Battey, 2007). These events have consequences on how students engage in sense-making. The critical events in our data led us to focus on the dynamic relations between tasks, questions, and prompts used by the teacher, as well as his actions. Consequently, we chose Watson’s analytical framework (2007) to understand the ways in which the teacher shaped the content of the lesson in terms of affordances and constraints.

Watson provides a coherent analytical framework for the examinations of the nature of mathematical activity in a classroom, allowing us to analyse what mathematics the teacher offers and what learners might perceive. She prompts us to focus specifically on the affordances and constraints to act mathematically, resulting from the choice of tasks, the questions and prompts forwarded by the teacher in a mathematics lesson. We interpret an affordance as an opportunity constructed by the teacher for students to make connections between concepts, as well as inferences, deductions or other possibilities of mathematical thinking. Contrary to common meaning where a constraint refers to a restraint, in Watson’s term, a constraint prompts the learner to opt for a particular interpretation at the expense of all the possibilities available. Watson presents seven overarching dimensions of mathematical orientations: (1) teacher’s elicitation of declarative/nominal/factual/technical statements; (2) learner’s expectation to exhibit certain actions; (3) teacher’s direction of learner’s perception/attention; (4) teacher’s prompting learner’s response; (5) discussion of implications (e.g., adapting procedures); (6) integration and connection of mathematical ideas; and (7) affirmation (e.g., application to other contexts). Each dimension contains a range of mathematical public tasks/prompts (refer to Watson, p.119 for a detailed description of the framework). We illustrate the relevant constructs as we present the data in Figures 2 and 3.

**Method**

*Setting and participants*

This study involved an introductory lesson on the teaching of fractions in a Grade 7 class (11-12 aged students) at an urban Indonesian school. The 35 students were considered to be above average ability for the local area. They had prior experiences working with fractions in primary school in terms of fraction identification, representation, comparison and simple fraction problem solving in involving arithmetic operations. The mathematics teacher had over 20 years of teaching experience and taught mathematics across the curriculum.

The current lesson was very interactive, demonstrated by the consistent questions and prompts used by the teacher and the resulting responses from students. Further, there was no focus on textbooks. This is in contrast to the recent World Bank video study (World Bank, 2010) that characterized Indonesian mathematics teaching as involving a high level of exposition driven by textbooks and reported that students had limited engagement in mathematical activities or in asking questions.

*Analysis of data*

The 45-minute, videotaped lesson was transcribed. To ensure the credibility of the analysis, two researchers independently examined the transcripts and noted where there was a clear shift in attention to particular mathematical objects (e.g., defining or visualizing fractions). Both then agreed to divide the transcripts into 35 events based on
the shift in the lesson orientation. Each event was coded individually and differences were resolved through discussions.

Each event focuses on a particular mathematical object (Watson, 2007), such as visualizing a half, making connections between the symbolic and pictorial form of a fraction, defining a term, and identifying the numerator and denominator of a fraction. Each event was coded based on Watson’s analytical framework, as illustrated in Figures 2 and 3.

**The fraction lesson**

The objective of the teacher’s lesson plan was to help his students understand the meaning of fractions (in terms of numerator and denominator), types of fractions (proper, improper; mixed numbers) and equivalent fractions, and identify the numerator and denominator of symbolic algebraic fractions.

The teacher started the lesson by asking the students to define a fraction. A student responded by saying: “a fraction is a whole object that is divided into several parts”. The teacher revoiced the idea without any further discussion and directed them to a paper-folding activity to represent halves and quarters. It was apparent that the teacher consciously engaged his students into sense-making, including transforming the concrete and pictorial representation of fractions into symbolic forms (1/2, 1/4, 2/4, 2/3), identifying the numerator and denominator and justifying their answers. The next task in the lesson was to prompt the students to make the link between the pictorial and symbolic representations of half and quarter. The teacher drew the students’ attention to the pictorial representation of the two fractions on the board:

T: Can you compare this picture [representing 1/2] and this picture [representing 1/4]? (Figure 1)

![Figure 1. Representation of 1/2 and 1/4](image)

S1: The top is divided into two and the bottom is divided into four.
T: Then, what is the relationship between these pictures [points to the pictures] and these fractions [points to the symbols, 1/2 and 1/4].
S1: Their relationship is: in both [pictures], one part is shaded.
T: What about the others?
S2: Both are divided.
T: Why is “one over two” here and “one over four” here [asking while pointing]? Where do the 2 and 4 come from?
S2: Because… [inaudible].

The transcript shows that the teacher chose two familiar fractions to prompt his students to see the relationship between the total number of parts and the denominator of the fraction, and the number of shaded parts and the numerator of the fraction. In this case, the affordances of this situation can be seen as all the possibilities that the task offers to students who are interacting with the mathematical objects (e.g., seeing the
pictorial and symbolic representations, one part is shaded in each picture). The constraint engendered by the teacher was to prompt students to infer the connection between the pictures and symbols. The next turn of the lesson involved the representations of improper fractions $1 \frac{1}{2}$. The teacher represented $1 \frac{1}{2}$ as shown in Figure 2.

<table>
<thead>
<tr>
<th>Constructs</th>
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<th>Students' actions</th>
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<tbody>
<tr>
<td>Teacher directs learners’ attention to multiple objects i.e., pictures.</td>
<td>T: Observe these two pictures! [...] What fraction does this picture represent?</td>
<td>S3: Two.</td>
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<tr>
<td>Teacher directs learners’ attention to the object which is perceived as having a single feature (a pictorial representation of an improper fraction $1 \frac{1}{2}$)</td>
<td>T: How many fraction is this? [points to the pictures]</td>
<td>S3: That is one fraction! [...] Try again. Come on!</td>
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<tr>
<td>And ask for learners’ informal induction.</td>
<td>T: Now it’s three per four [= three quarters] [points at $\frac{3}{4}$], where does the three come from?</td>
<td>S4: [She writes $\frac{3}{4}$ by adding its numerators and denominators]</td>
</tr>
<tr>
<td>T: Then where does the four come from?</td>
<td>S5: The sum of all the parts Sir, the parts that were divided.</td>
<td></td>
</tr>
<tr>
<td>Teacher directs learners’ attention to identify the features.</td>
<td>T: This is divided by? [points at the circle that is divided into two and all its parts are shaded in]</td>
<td>S5: Two plus two equals four.</td>
</tr>
<tr>
<td>T: Then, this is also divided by two. Then? [points at the circle that is divided into two, but only one part is shaded in]</td>
<td>S5: Add them together Sir, two plus two equals four.</td>
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<tr>
<td>T: How many circles are here? [teacher points at the two circles]</td>
<td>S5: It can’t.</td>
<td></td>
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<tr>
<td>T: There are two, right? How many circles are here? [points at a circle]</td>
<td>S5: Divided by two.</td>
<td></td>
</tr>
<tr>
<td>T: It’s divided by two, right?</td>
<td>S5: Two plus two equals four.</td>
<td></td>
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<tr>
<td>T: This is one divided by... [points at a circle] two.</td>
<td>S6: [No response from the students]</td>
<td></td>
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<tr>
<td>T: So if it’s three over four [= three quarters]?</td>
<td>S7: Which one’s correct?</td>
<td></td>
</tr>
<tr>
<td>T: Where is the two from?</td>
<td>S5: We don’t know yet.</td>
<td></td>
</tr>
<tr>
<td>T: Then?</td>
<td>S5: Two circles Sir.</td>
<td></td>
</tr>
<tr>
<td>T: Maybe someone has another opinion? Any other opinions?</td>
<td>S6: Then, the three from the shaded parts.</td>
<td></td>
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<tr>
<td>T: The correct answer is still hidden. Do you have any other ideas. Come on! Who has another opinion?</td>
<td>S7: Which one’s correct?</td>
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Figure 2. Snapshot: Teacher and students’ action in interpreting $1 \frac{1}{2}$
The predominant part-whole conception of fractions driving the discussion led to multiple unpredictable constraints. For the teacher, the picture drawn in Figure 2 represented one and a half as one of the circles was taken as a unit. However, the students did not appear to interpret the picture from this perspective. It is more likely that they translated the part-whole understanding of the proper fraction in the previous examples to this new situation. The line segment in the first circle may have prompted the student (S3) to infer that the shaded parts in the first circle represented 2/2 rather than one whole. The two circles may afford interpretations in terms of one object (as a unit) or as two separate objects. Student S3 was prompted by the teacher’s comment to consider the two pictures as one fraction and therefore she asked whether she had to add the two fractions. She then added the numerators and denominators of the two fractions, 2/2 + 1/2 = 3/4. On the other hand the pictures in Figure 2 afforded another interpretation for student S4. She saw 2 circles, 3 shaded parts and 4 equal parts and gave the answer 2 3/4, and was supported by student S5.

The teacher’s attempt to resolve the issue
To further assist students to interpret mixed numbers using an area model, the teacher added another example, representing 1 1/4. The transcript in Figure 3 shows the ongoing confusion, biased by the prevailing part-whole conception of fraction.

Again the representation made by the teacher provided unintended affordances. Student S8 focused on the shaded and unshaded parts as well as the two wholes to give the answer 2 5/3. Similarly, student S9 focused on the shaded and unshaded parts. The enduring part-whole conception was still apparent at this point of the lesson. The pictorial representation of the first square (subdivided into four and shaded) prompted student S9 to represent it as 4/4 rather than one whole. It is only when the teacher asked them “what is four divided by four,” that a student said it was 1, and then changed 2 1/4 to 1 1/4.

Given space limitations, we have only reported on selected events. However, in the remaining part of the lesson, the teacher prompted the students to discuss the different types of fractions. Then they were asked to solve a set of three tasks involving the symbolic representation of fractions from given pictures, identification of the numerator and denominator of symbolic fractions ((a) 5/7; (b) 17/18; (c) 5p/q; (d) x+5/y+z), and finding one fraction between two given fractions ((a) ¼ and ¾; (b) 1/3 and 2/3; (c) 3/8 and 5/12). The students experienced much difficulty in finding a fraction between two given fractions. This led the teacher to bring forth the notion of equivalent fractions and the representation of fractions using number lines.

Discussion
What opportunities to act mathematically are afforded and constrained by tasks, questions, prompts and teacher actions in the teaching of fractions? The part-whole conception of fractions that the teacher prioritized led the students to interpret fractions merely on the basis of shaded and unshaded regions. Although the students wrote 4/4 and 1/4 in Figure 3, they gave the answer 2 1/4. This points to the possibility that the 4/4 did not necessarily represent one whole for them but was just representing 4 shaded parts out of 4 parts. The students were not adequately prompted to see the two parts (circles and squares) in Figures 2 and 3 as one thing, one fraction. The students explicitly expressed that the two pictures represented two fractions. In both figures, the
teacher subdivided the whole that proved to be more constraining than enabling for his students.

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<td>Teacher directs learner’s attention to an object which is perceived as having a single feature</td>
<td>T: This is divided into four (points at the first square). This is also divided into four (points at the second square). Now I’ll shade this (shades in the first square) and this (shades in the second square) like this. Who can represent this in fraction? [Teacher talks while drawing the representation of 1 1/4 as below]</td>
<td>[A student comes up to the board and he writes the fraction 2 5/3 below the squares drawn by the teacher]</td>
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<td></td>
<td>[Figure 3: Snapshot: Teacher and students’ action in interpreting 1 1/4]</td>
<td></td>
</tr>
<tr>
<td>Teacher directs learner’s attention to identify the relationship between the visual and symbolic form.</td>
<td>T: Where is the two from?</td>
<td>S8: Two squares.</td>
</tr>
<tr>
<td></td>
<td>T: Then where is the five from? Where was the five from?</td>
<td>S8: From the shaded ones.</td>
</tr>
<tr>
<td></td>
<td>T: Where is the three from?</td>
<td>S8: The ones those aren’t shaded in.</td>
</tr>
<tr>
<td>Teacher asks for learners’ response to make informal induction.</td>
<td>T: Everyone else, we have two squares, the squares are divided into? One is divided into? Four. And what is meant by this... Come on! Anyone else please.</td>
<td>S9: [A student comes forward and writes the fraction 4/4 below the first square and ¼ below the second square, then they are added to give 2 1/4.]</td>
</tr>
<tr>
<td></td>
<td>T: Is this right? Is it already correct?</td>
<td>Ss: Yes.</td>
</tr>
<tr>
<td>Teacher directs learner’s attention to an object which is perceived as having a single feature</td>
<td>T: This is four over four (points at square 1). Please explain (asks the student who wrote it). Explain. Where is the four over four from? Where is the four over four from? This is one over four, which is right. Now is this right? [points at the sum of 4/4 and ¼ from the student]?</td>
<td>Ss: Yes.</td>
</tr>
<tr>
<td></td>
<td>T: What is four divided by four?</td>
<td>S4: One. [The student then changes 2 into 1 in the fraction 2 ¼ which then became 1 ¼].</td>
</tr>
<tr>
<td></td>
<td>T: Is this right? [asks the whole class]</td>
<td>Ss: Yes.</td>
</tr>
<tr>
<td></td>
<td>Do you agree?</td>
<td>Ss: Yes.</td>
</tr>
</tbody>
</table>

*The conversation above was captured from the transcript Line 345-397.*

One of the main findings in this study is that it shows how the teacher’s emphasis on the part-whole definition of fractions led to a tunnel-vision conception of fraction and caused challenges for students to act mathematically. The teacher attempted to use the part-whole model to build the notion of improper fractions. Consequently, the students
experienced much difficulty in interpreting a fraction greater than one as a number in its own right. Rather, the diagrams representing the mixed number (Figures 2 and 3) were interpreted as two fractions. They could not see the multiplicative relationship between a unit fraction, such as one-fourth, which can be iterated five times to produce five-fourths (Figure 3). This shows that the tasks chosen by the teacher and its associated conceptual representation influenced the sense that the students made of the mathematical concept. This teaching event illustrates an instance of a teacher-generated constraint in that it narrowed the students’ attention on one aspect of fraction, namely the part-whole meaning. Had the teacher used a conceptually different task, the affordance that it would have generated might have been totally different. For example, the teacher could have drawn a rectangle without sub-division on the board and asked the students to draw seven-fifths of it, or the teacher could have shown an orange plus a half orange.

Before he asked the students to interpret the mixed number, the teacher had already divided the whole into 2 (Figure 2) and into 4 (Figure 3). This splitting action may have prompted the students to think in terms of halves in the first case and quarters in the second case. Observing the challenges that the students experienced in the snapshot in Figure 2, he made the decision to use a second task to attempt to resolve it (Figure 3). However, this additional task did not resolve the issue, as the underlying conceptual model (area model) was the same. This teaching event exemplifies another example of a constraint as the teacher focused the students’ attention towards a particular way of looking at the problem.

Our data analysis shows that the teacher intended to make the lesson mathematically engaging. He provided a number of affordances in terms of the multiple representations of fractions (symbols, words, pictures, paper-folding activity). Further, his worksheet contained a set of rich tasks for an introductory lesson on fractions. However, the teacher did not discuss the unit of fraction, which was one of the critical features of the fractional concept.

Conclusion
Although we snapshot only one lesson, the findings from this study are revealing. First, it shows how a pictorial representation of a fraction can provide a multitude of affordances, resulting in correct or incorrect responses. Second, the findings show that the representation of 2/2 and 4/4 may not necessarily represent one whole for students. Third, the study shows how the part-whole conception of fractions tends to skew students’ interpretation of mixed numbers presented in pictorial form. Fourth, this study highlights how public discussions tend to influence students’ interpretation of fractions in the absence of a teacher’s directions. The students simply extended their incorrect deduction to the next example provided by the teacher. Finally, the finding shows how a teacher’s intention in using a drawn representation may differ from students’ interpretation. In this study, the experienced teacher made visible effort to prompt his students to make sense of improper fractions. The inadequate attention of the teacher to the multiple features entailed in the representation of fractions (Petit, Laird, & Marsden, 2010) and the absence of explicit explanations on the underlying ideas such as unit and unitizing (Lamon, 2006) hindered students’ ability to develop their understanding of fractions.
References

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