Examination of the issues related to the teaching/learning of the normal distribution curve at high school by using CAS calculator

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I. Introduction

From which function could the probability density function $f(x)=\frac{1}{s\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-m}{s})^2}$ (the constants $m, s\ (s \neq 0)$) of the normal distribution in the field of statistics at the high school be induced? What historical developmental background does the function have? If the function was induced from $f(x)=e^{-x^2}$, what kind of role do $\sqrt{2\pi}$, $\frac{1}{2}$ and the constants $m, s\ (s \neq 0)$ play? Which process or procedure does the area of the normal distribution curve go through? This study was carried out with the purpose of solving the question related to the normal distribution.

According to 'the Basics of Calculus and Statistics' and 'Integral Calculus and Statistics' in the Korean secondary math curriculum, there is still a great level of emphasis on the written environment just like the past. As a result, instead of the understanding based on the examination of the properties and relationships of the probability distribution, the method of using established facts is still greatly emphasized. Since it is actually impossible to identify various properties and relationships of the probability distribution in the written environment, it is absolutely necessary to utilize the engineering part efficiently in order to easily examine, identify and understand such factors. In this study, such problems were to be solved in terms of the engineering aspect based on the utilization of CAS calculator.

It has already been found that CAS calculator, which has the function of CAS (Computer Algebra Systems) has a positive influence on the teaching/learning of mathematics at school, while providing strong points and structural changes in the process of mathematical education. Also, there are various great changes related to the teaching/learning method (Heid, 2003; Lagrange, 2005; Tall, Smith, & Piez, 2008). In particular, Kutzler said that CAS, as a mathematical tool for the teaching/learning of mathematics, could play the roles of four tools such as those of trivialization, experimentation, visualization and concentration (Kutzler, 2003)

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For the examination of various properties of the normal distribution, this study focuses on the utilization of CAS calculator, because it is impossible to accomplish the goals (the approximation of the binominal distribution to the normal distribution, the curve examination of $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, the area examination of the normal distribution curve based on the linear changes of the normal distribution function, and the examination of the normal distribution curve in various forms) suggested in the curriculum in a written form in terms of such tool functions as the trivialization, experimentation, visualization and concentration of CAS calculator. Therefore, in this study, CAS calculator was used to examine various properties of the normal distribution curve. From such a process and the related results, the teaching issues about the normal distribution could be induced.

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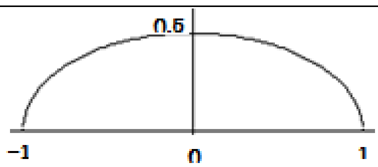
II. Historical developmental background of the normal distribution curve

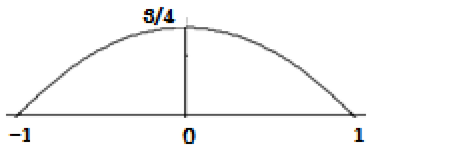
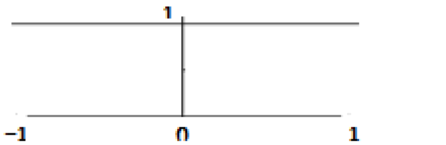
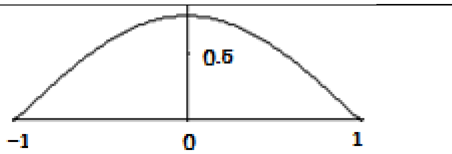
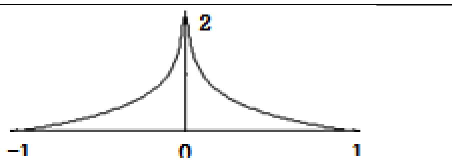
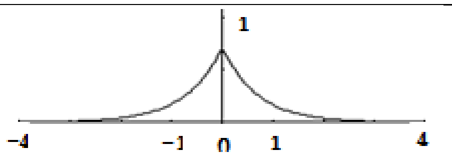
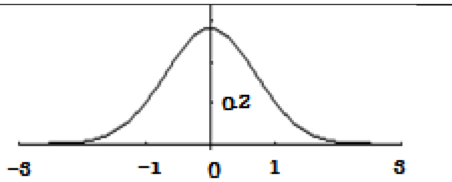
The distribution concept appeared in the Theory of Errors in the 18th century. At that time, people argued about whether the arithmetic average could be used to reduce errors or not. Since it was impossible to make a judgment about individual errors, the relationship among the errors was to be identified. Simpson was the first person who actually carried out such a converting process. In 1756, Simpson used the simple probability function in order to argue that the average of the observed values was more appropriate than single observed values through several observations, identifying the relationship among the errors of the observed values (Bakker, 2004).

In terms of mathematics, the normal distribution can be expressed in various ways. The first historical and typical method is to find the normal distribution from the limit value of the binominal distribution $bin(n,p)$, $n \to \infty$. This is the result of the limit theorem proposed by De Moivre-Laplace. Based on the Central Limit Theorem, it is said that the sum of the independent probability variables can be approximated to the normal distribution. For example, such personal key data as the parents' heights, the dietary habits and the quantity of motion could influence one's height. Even if such factors do not follow the normal distribution, their sum could be approximated to the normal distribution. As a result, a number of phenomena could be explained by the normal distribution (Wilensky, 1997).

The normal distribution and the normal distribution curve could be called as the law of errors, the law of frequency, the Gaussian Curve and the Laplace-Gauss Curve. Among the names that are not mentioned here, there is 'the De Moivre Distribution' by Freudenthal, since De Moivre defined the function for the first time. The research results by Gauss for the normal distribution curve were used in various fields such as astronomy, social science and anthropology. The following figure focuses on the historical development process of the normal distribution curve in terms of graphs and functions.

Table II-1. Historical developmental process of the normal distribution curve (Bakker, 2004)

Shape of the Graph	**Functional Formula**
	Lambert (1765): Half Circle $f(x)=\frac{1}{2}\sqrt{(1-x^2)}$

	Lagrange (1776): Consecutive Quadratic Function $f(x)=\frac{3}{4}(1-x^2),\ -1\le x\le 1$
	Lagrange (1776): Consecutive Equality Function
	Lagrange (1781): Cosine Function $f(x)=\frac{\pi}{4}\cos(\frac{\pi x}{2}),\ -1\le x\le 1$
	Laplace (1781): Log Function $f(x)=\frac{1}{2}\log\frac{1}{\lvert x\rvert},\ -1\le x\le 1$
	Laplace (1784): Double Exponential Function $f(x)=\frac{1}{\sqrt{2}}e^{-\sqrt{2}\lvert x\rvert}$
	Gauss (1809) & Laplace (1810): Error Law or Normal Distribution $f(x)=\frac{1}{\sqrt{\pi}}e^{-x^2}$

III. Property examination of the normal distribution curve

1. Form of $f(x)=e^{-x^2}$ which is the original shape of the normal distribution curve

The graph of the normal distribution is referred to as the normal curve. As mentioned before, it was studied by Gauss and Laplace in the 19th century for the first time. The original shape of the normal curve is based on $f(x)=e^{-x^2}$. Also, it can be induced from $f(x)=e^{x}$ and $f(x)=e^{x^2}$. Throughout this study, the normal curve is referred to as 'the normal distribution curve' in accordance with the current high-school math curriculum, while the function $f(x)=e^{-x^2}$ is referred to as 'the original shape of the normal distribution curve'.

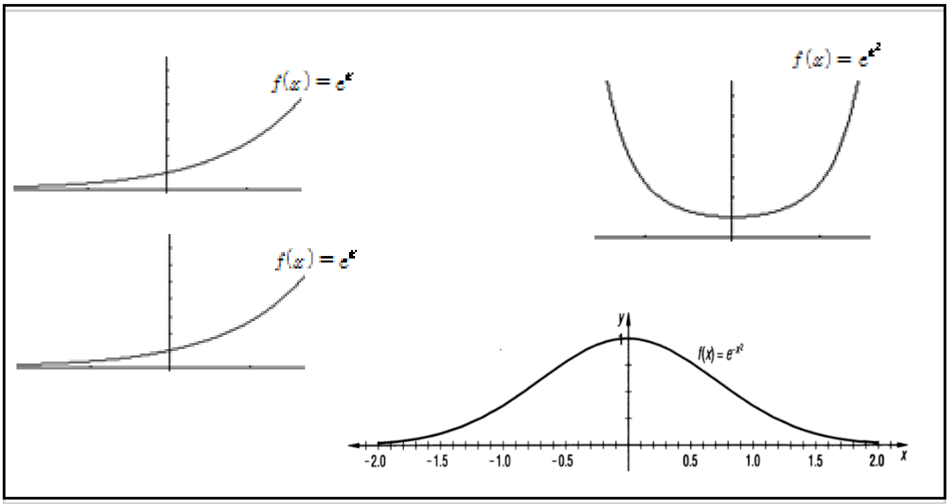


Figure III-1. Original shape of the normal distribution curve

2. Property examination of the original shape of the normal distribution curve

The following properties can be obtained from the original shape of the normal distribution curve $f(x) = e^{-x^2}$.

First, $(0,1)$ is the maximum value, while the y intercept is 1.

Second, since $f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$, the y-axis symmetry is applied.

Third, the x-axis is an asymptotic curve. If $|x|$ increases, x^2 also increases. However, e^{-x^2} leads to $\frac{1}{e^{x^2}}$, which is the reciprocal of the big value. Therefore, as the value of x becomes greater, the value of $f(x) = e^{-x^2}$ becomes smaller.

Fourth, the curvature change of the curve is another property. In the graph, the curve becomes convex near the y intercept, while it becomes concave as it moves away from the y intercept.

As shown above, there are two inflection points in the graph, which are convex and concave. In order to find the inflection points, by examining $f''(x) = 0$, it is possible to get $x = \pm \frac{1}{\sqrt{2}} \approx \pm 0.707$. By calculating the y value of the two points, we could get $y \approx 0.61$. It is possible to get such a fact by considering the results of the CAS calculator below.

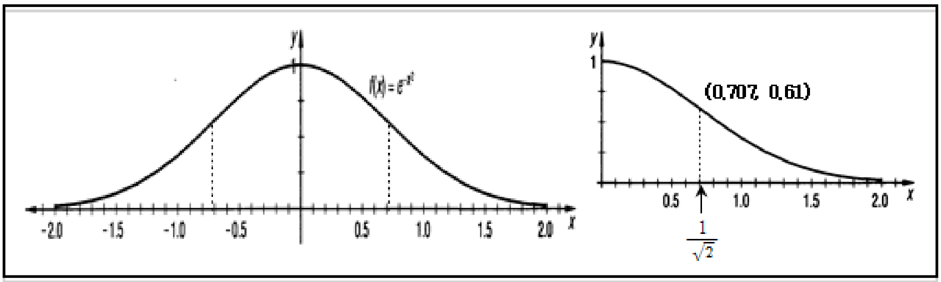


Figure III2. x-coordinate of the inflection point of $f(x) = e^{-x^2}$

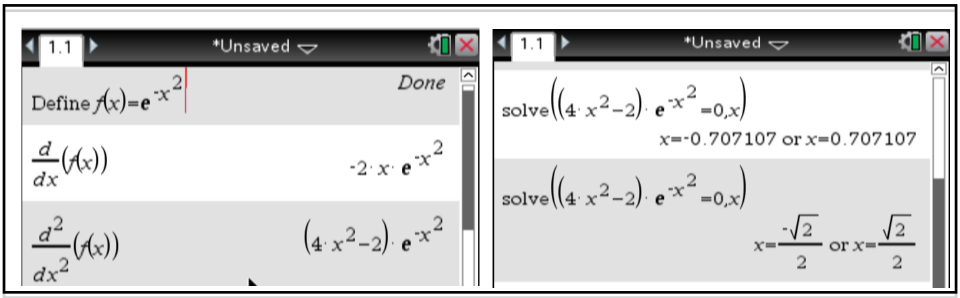


Figure III3. Results for the x-coordinate of the inflection point for $f(x) = e^{-x^2}$ *by using CAS calculator*

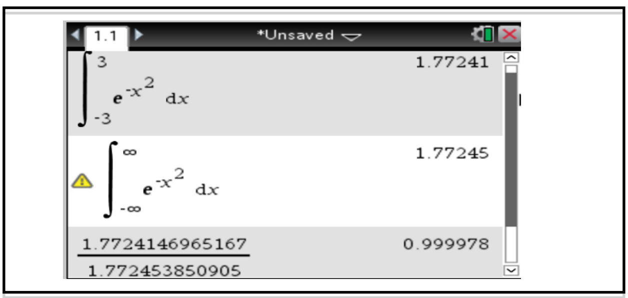


Figure III4. Calculation of the area between the graph of $f(x) = e^{-x^2}$ *and the x-axis by using CAS calculator*

IV. Linear changes of the normal distribution curve

By applying linear changes to $f(x) = e^{-x^2}$ which is the original form of the normal distribution curve, the function can be changed to $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ which is the probability density function of the normal distribution curve. Also, based on the results, it is possible to extract important properties for the standard normal distribution curve.

1. Conversion of $x=\pm\frac{1}{\sqrt{2}}$ *which is the inflection point of the normal distribution curve to* $x=\pm 1$

In order to change $\pm\frac{1}{\sqrt{2}}$ which is the inflection point of $f(x)=e^{-x^2}$ to 1, the value of the x-coordinate must be estimated based on $\sqrt{2}\times x$. The conversion can be expressed in the equation of $S_1:(x,y)\rightarrow(\sqrt{2}x,y)$. Therefore, the conversion can be applied as follows.

Since $x'=\sqrt{2}x$, $x=\frac{1}{\sqrt{2}}x'$.
Since $y'=y$, $y=y'$.
$y'=e^{-(\frac{1}{\sqrt{2}}x')^2}=e^{-\frac{(x')^2}{2}}$
$\therefore y=e^{-\frac{x^2}{2}}$

By drawing the graphs of the converted function of $y=e^{-\frac{x^2}{2}}$ and the original function of $y=e^{-x^2}$ together, the following graph can be established with the invariable maximum value of 1. However, after getting the function and calculating the inflection point, it is possible to see that the location of the inflection point has been changed to $x=\pm 1$ and the curve has been moved up towards the direction of the y-axis from the inflection point. As a result, the area under the curve has been enlarged.

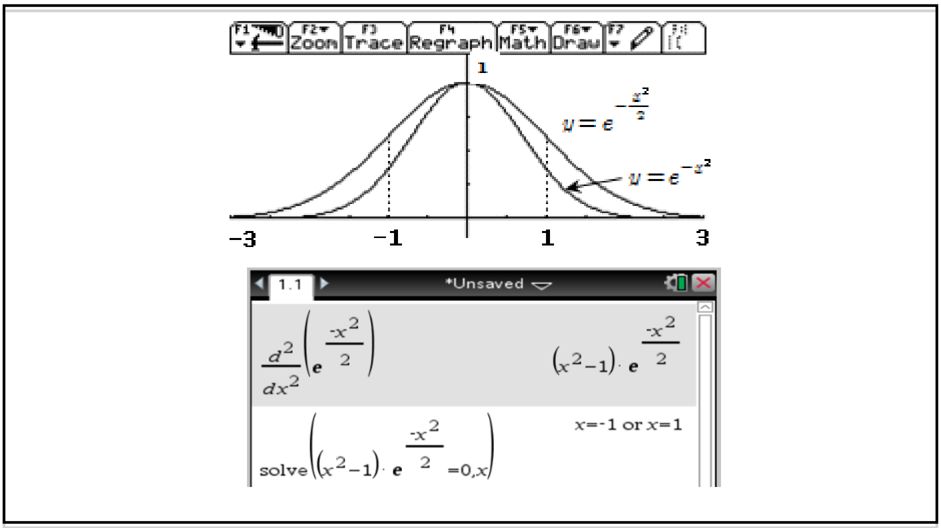


Figure IV1. The graph of $y = e^{-\frac{x^2}{2}}$ *based on the conversion of the inflection point and the results given by CAS calculator*

2. Conversion of the area of the normal distribution curve to 1

As mentioned before, the area under the curve of $f(x) = e^{-x^2}$ is $\int_{-\infty}^{\infty} (e^{-x^2})\,dx \approx \sqrt{\pi}$, while the area under the curve of $f(x) = e^{-\frac{x^2}{2}}$ has become bigger than $\sqrt{\pi}$ due to the conversion of S_1. Then, what would be the resulting area? By following the process of $S : (x, y) \rightarrow (ax, ay)$, the converted area is $|ab|$ times of the original area, becoming | ab | x (Original Area) according to the Area Scale Change Theorem. However, according to the conversion of $S_1 : (x, y) \rightarrow (\sqrt{2}\,x, y)$ and the scale, the area of $f(x) = e^{-\frac{x^2}{2}}$ is $|\sqrt{2} \cdot 1| \times \sqrt{\pi} = \sqrt{2\pi}$. Based on the movement up towards the y-axis from the inflection point of $x = \pm 1$, the area has been enlarged from $\sqrt{\pi}$ to $\sqrt{2\pi}$. Therefore, in order to make the area become 1, the scale needs to be $S_2 : (x, y) \rightarrow (x, \frac{y}{\sqrt{2\pi}})$. Then,

Since $x' = x$, $x = x'$.

Since $y' = \frac{1}{\sqrt{2\pi}}\,y$, $y = \sqrt{2\pi}\,y'$.

$$\sqrt{2\pi}\,y' = e^{-\frac{x^2}{2}}$$

$\therefore y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

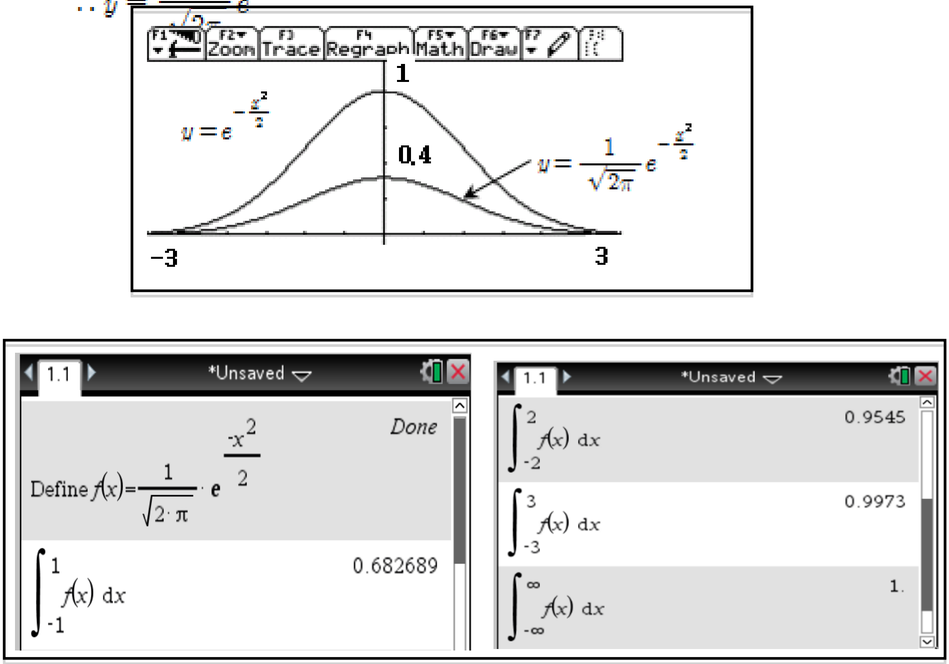


Figure IV2. The graph of $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ *based on the conversion of the area and the results given by CAS calculator*

By looking at the above graph and the results given by the CAS calculator for the integrated calculus, the standard normal distribution curve of $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ can have the following properties.

First, the area under the x-axis and the curve becomes 1.
Second, the inflection point is applied in case of $x = \pm 1$, becoming convex in the range of $-1 < x < 1$ and concave under the condition of $|x| > 1$.
Third, since the function is even, the y-axis symmetry with $x = 0$ is applied.
Fourth, the maximum value of the function is $f(0) = \frac{1}{\sqrt{2\pi}} \approx 0.3989$.
Fifth, the equation of $f(x) > 0$ is applied for every real number.
Sixth, in case of $x \to \pm\infty$, the equation of $f(x) \to 0$ is applied.

Then, by considering the results of the above integral calculus, the area in the range of $-1 < x < 1$ is 0.683, which is subject to 68.3% of the entire area. The area in the range of $-2 < x < 2$ is 0.954, which is subject to 95.4% of the entire area. Also, the area in the range of $-3 < x < 3$ is 0.997, which is subject to 99.7% of the entire area.

When the normal distribution is shown as $N(m, \sigma^2)$, the value of the probability $P(a \leq X \leq b)$ for the probability variable X based on the normal distribution is the same as the area between the graph of the normal distribution curve and the x-axis. Also, the value of the probability $P(m - k\sigma \leq X \leq m + k\sigma)$ $(k = 1, 2, 3)$ can be estimated as follows based on the current results.

$$P(m-\sigma \leq X \leq m+\sigma) = P(m-\sigma < X < m+\sigma) \fallingdotseq 0.683$$
$$P(m-2\sigma \leq X \leq m+2\sigma) = P(m-2\sigma < X < m+2\sigma) \fallingdotseq 0.954$$
$$P(m-3\sigma \leq X \leq m+3\sigma) = P(m-3\sigma < X < m+3\sigma) \fallingdotseq 0.997$$

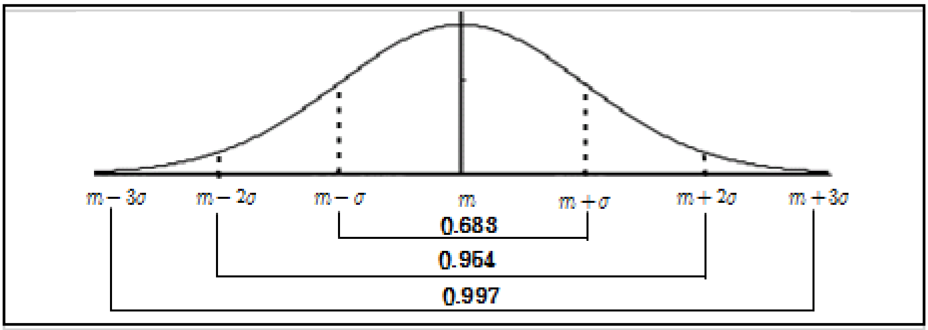


Figure IV-3. The relationships among the average m of the normal distribution, the standard deviation σ and the area of the normal distribution curve

Generally, the x-axis of the probability density function of the standard normal distribution is changed to the z-axis. Therefore, the functions showing the standard normal distribution can be expressed as follows. Also, the axis of the graph is changed to the z-axis ($-\infty < z < \infty$) and the $f(z)$-axis.

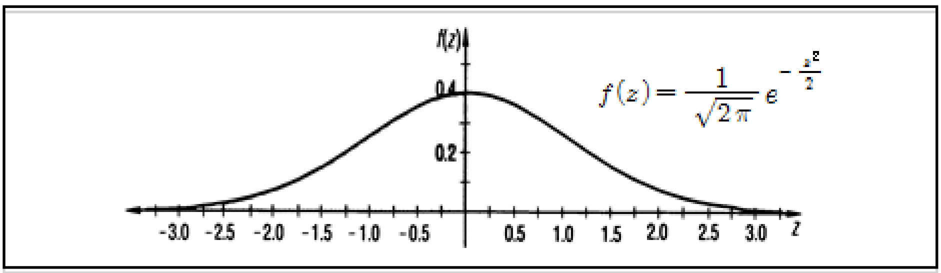


Figure IV-4. Standard normal distribution curve based on the probability variable of z

3. Various types of normal distribution curves and properties

Even if optional linear and scale changes are applied to the standard normal distribution curve, the result is still the normal distribution curve. In particular, when m and s are constants and the equation of $s \neq 0$ is applied, the graph based on the following equation becomes the normal distribution curve with the size under the curve being 1.

$$f(x) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-m}{s})^2}$$

Therefore, such a curve could show the probability distribution. The graph which changes m and s regarding such a probability density function is shown in Figure Ⅳ-5. From such a figure, it is possible to find important properties.

First, when the equation of $m > 0$ is applied in case of the probability density function, the graph shows the parallel translation towards the right. In case of $m < 0$, the graph shows the parallel translation towards the left. Also, the shape of the graph shows no change.

Second, s refers to the level of diffusion for the curve. As s becomes bigger, the level of diffusion for the curve also becomes bigger. When it becomes smaller, the level of diffusion of the curve becomes smaller and gathers around the center.

Third, the curve is symmetrical to the straight light with $x = m$.

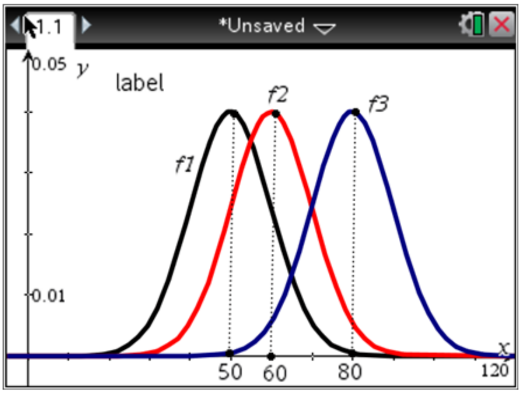
Also, in such a case, m shows the average of the probability distribution, while s shows the standard deviation. In 1730s, Abraham DeMoivre found the following equations with the related curves which were subject to the normal distribution with the temporary average of μ and the standard deviation of σ, completing a perfect mathematical equation for the normal distribution. As mentioned before, in regard to the equation, the constant of $\frac{1}{\sigma\sqrt{2\pi}}$ was applied to make the size of the area under the curve become 1.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

$$f1(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-50}{10}\right)^2}$$

$$f2(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-60}{10}\right)^2}$$

$$f3(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-80}{10}\right)^2}$$



V. Conclusion

This study focuses on the utilization and examination of various properties of the probability density functions of the normal distribution curve and the normal distribution, which are suggested in the chapter of the probability distribution in the field of statistics at the high school by using CAS calculator, while drawing the issues for the teaching/learning of the normal distribution curve. In order to achieve such a purpose, a CAS calculator was used for the examination process mainly based on the approximation of the binominal distribution to the normal distribution, the examination of the normal distribution curve, the examination of the size of the normal distribution curve based on the linear changes of the normal distribution curve, and the examination of various types of normal distribution curves. Based on the results of the study, the following issues were found in the teaching/learning of the normal distribution curve.

$$f1(x) = \frac{1}{20\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-50}{20}\right)^2}$$

$$f2(x) = \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-50}{5}\right)^2}$$

$$f3(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-50}{10}\right)^2}$$

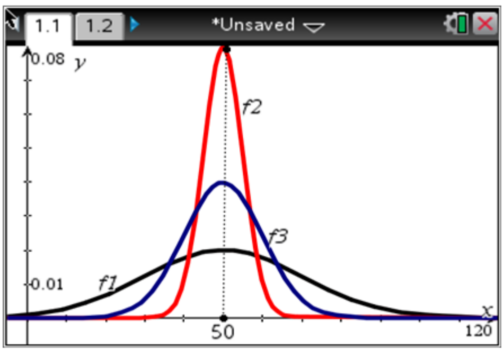


Figure Ⅳ-5. Various Changes for the Shape of the Normal Distribution Curve based on the Constants of m and s ($s \neq 0$)

First, since it is impossible to help students understand the fact that the binominal distribution is approximated to the normal distribution by suggesting it in a figure as shown in the textbook, it is necessary to provide chances for experiments regarding the utilization of such engineering devices as CAS calculator.

Second, according to the current high-school math curriculum, it is impossible to estimate the size of the area under the x-axis and the normal distribution curve before learning the integral calculus. Therefore, by using CAS calculator, it is possible to calculate the size based on the probability, while increasing the level of recognition for the connectivity among mathematical contents.

Third, according to the current high-school math curriculum, because of the fear caused by the complex functions based on the advanced introduction of the linearly-changed function $f(x)=\frac{1}{s\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-m}{s})^2}$ with such constants as m, s $(s \neq 0)$, it is difficult for students to understand the distribution curve. Instead of such an introduction method, it would be better to introduce the linear changes from $f(x)=e^{-x^2}$, which is the original form of the normal distribution curve, to $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ in order to decrease the level of burden for the complex functional formula and increase the level of understanding for the functional forms.

Fourth, according to the current high-school math curriculum, the normal distribution of $f(x)=\frac{1}{s\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-m}{s})^2}$ is introduced first before $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is introduced as the standard normal distribution. As mentioned before, it is possible to prevent students from understanding the concepts of the functional formulas by introducing the functional formulas based on the linear changes first. Therefore, as suggested in this study, it would be better to introduce the standard normal distribution curve first in order to decrease the level of burden experienced by students.

Fifth, according to the current high-school math curriculum, the functional formula of the normal distribution curve is given as $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-m)^2}{2\sigma^2}}$. By looking at the functional formula of the normal distribution curve, $x-m$ is the deviation which shows the distance and direction from the average. Therefore, if the value is raised to the second power, it becomes the square deviation. It is useless to divide the value with the variance σ^2 which is the square of the standard deviation. Therefore, in order to express the process of dividing the distance and direction from the average with the standard deviation and estimating the unit length, it would be more appropriate to use the equation $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$.

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