Conceptual understanding of proportional reasoning via poster proofs in teacher professional development

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Introduction

Proportional reasoning is fundamental to many important mathematical concepts and is often regarded as the pathway to performing well in algebra. Students use proportional reasoning in early math learning, for example, when they think of the number 10 as two fives or five twos rather than thinking of it as one more than nine (Suh & Seshaiyer, 2012). This is an essential developmental step for students to transition from additive to multiplicative reasoning (Sowder et. al., 1998). Although additive reasoning can develop intuitively, multiplicative (proportional) reasoning is difficult for students to develop and often requires formal instruction. It requires reasoning about several ideas or quantities simultaneously. It requires thinking about situations in relative rather than absolute terms. This ability to think and reason proportionally is very important in the development of a student's ability to understand and apply mathematics. It is essential for mathematics teachers therefore to (a) understand how students develop multiplicative (proportional) reasoning, (b) build on students' prior concepts such as multiplication and division of whole numbers to strengthen students' proportional reasoning, and (c) develop learning environments, contexts and experiences for students that encourage multiplicative comparisons to prepare them for higher-level mathematics topics involving proportional reasoning.

Research has shown that a content-focused professional development (PD) leads to improvements in teacher content knowledge with a focus on student learning goals, highlighted concepts to be addressed, their development over time, difficulties students may encounter, and monitoring student understanding (Suh and Seshaiyer, 2013, 2014). Sztajn (2011) reports that mathematics PD is an emerging research field that needs high quality reports on description and a standard for reporting, including design decisions. As designers and researchers, we were intentional in our design decisions with the goals of developing teachers' specialized knowledge for proportional reasoning. To evaluate the collaborative nature of designing PD, Suh and Seshaiyer (2014) used the collective self-study method to examine how purposively designed experiences such as the content-focused institute in the summer with school-based follow-up Lesson Study cycles in the fall encouraged vertical articulation of algebraic connections.

In this work, we present one such PD program that systematically introduces the concept of proportional reasoning through problem based learning activities. Specifically, the paper presents how the pedagogical practices of a group of 85 elementary and middle grades teachers that participated in a summer institute were impacted in proportional reasoning through problem solving activities. The observations on their understanding of proportional reasoning through these problems, their reflections on the topic, as well as misconceptions found in the research are presented.

Context of the study

The PD program included a content-focused summer institute and a follow-up Lesson Study throughout the academic year focused on engaging teachers in active learning through rational numbers and proportional reasoning tasks, exploring pedagogical strategies, utilizing mathematics tools and technology, and promoting connections aligned and coherent to the elementary and middle school curricula. Daily activities in the summer institute included modeled lessons using a variety of mathematics tools and technology and in-depth conversation about the proportional reasoning, pedagogical strategies such as using problem solving, and multiple representations. The study presents the analysis of teacher work through poster proofs, as well as follow-up discussions to help the instructors understand the level of conceptual thinking teachers used in proportional reasoning to solve the problem, rather than a traditional approach.

For the week's activities, teachers were grouped by county and wore color-coded nametags. Four large, adjacent conference rooms were utilized. A large meeting area was centrally located. Teachers were assigned to a homeroom (Room B, C, D, or E). Each room was furnished with four long rectangular tables at which six to eight teachers were seated. For the morning activities, they reported to the same room each day while the instructors moved to a different room each day. In this way, each group saw each of the four instructors, but on different days. Additionally, teachers were also randomly assigned to afternoon groups. These groups attended workshops and had the opportunity to interact with teachers from different counties.

The teachers were also expected to maintain a journal where they would reflect on the problems they worked on during the day. The daily topics included reasoning up and down, direct and inverse thinking, unitizing, and ratios and proportional thinking. For this study, we decided to focus on one specific problem that was provided to the teachers as an opening problem, the cathedral problem that is adapted from S. Burns (2003): *While building a medieval cathedral, it cost 37 guilders to hire 4 artists and 3 stonemasons, or 33 guilders for 3 artists and 4 stonemasons. What would be the expense of just 1 of each worker?*

The next section includes the data analysis of teacher thinking that went into solving this problem. Each group was asked to create a poster representing their solution and each teacher was also asked to reflect on how they participated in the problem solving process and how they would take this problem back to their classroom to present to their students. Photographs of all of the posters created by the teacher groups of the cathedral problem were taken and arranged in order of the day of completion. Room E completed this problem first, followed by Rooms B, C, and D, respectively. The data from the posters were analyzed for content, connections between concepts, and any possible differences related to the time already spent in the seminar. Data was also analyzed from the teacher reflections for common themes as well as individual perspectives. We will present the analysis of the poster artifacts based on what the teachers in the respective groups shared followed by our analysis of the corresponding teacher reflections.

Data analysis through poster artifacts

Room E was the first homeroom to work on this problem. The first group, Figure 1, argued that if one artist and one stonemason together made $11, then the total for three of each would be $33. However, they reasoned that because we know that $33 is enough

to pay those six workers plus another stonemason, then one stonemason and one artist must together make less than $11. This group then used the guess-and-check method. They first assumed that the total for one artist and one stonemason was $8. They tried the combination of $1 for the cost of one artist and $7 for one stonemason ($8 total); however, they discovered that the total for 3 artists and 4 stonemasons was less than the needed $33. They tried other combinations but saw that the total cost was decreased; so, they abandoned the idea of an $8 total. They then assumed that the total for one artist and one stonemason was $9. However, their starting guess for the cost of one artist was $2, not $1. They found that the combination of three artists at $3 each and four stonemasons at $6 did total the needed $33. However, when they used these amounts in the second scenario, they found that it did not work: four artists at $3 each and three stonemasons at $6 did not total the needed 37 dollars. They then assumed that the total for one artist and one stonemason was $10. Using the same logic, starting at $1 per artist and $9 per stonemason, then $2 per artist and $8 per stonemason, etc., they arrived at a solution of $3 per artist and $7 per stonemason, which they demonstrated would satisfy both requirements. In all three *guess-and-check* calculations, they assumed that the artists would earn less than the stonemasons would; so, they arrived at the correct figures for the solutions but had the assignments to the two types of workers backwards. They showed that four artists at $3 each and three stonemasons at $7 each would total $33. However, the original question stated that the cost of $33 applied to three artists and four stonemasons. So, although their logic was correct, they made a minor error in the interpretation.

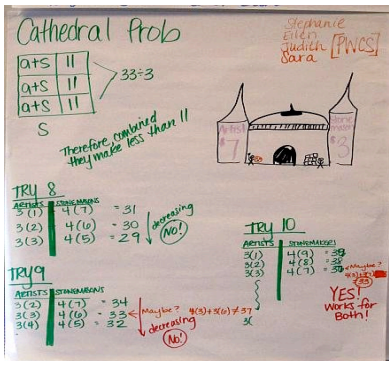

Figure 1. Guess and check

The second group, Figure 2, presented *tabular and pictorial* representations of the two scenarios. There are also several indications that they chose the values of seven and three dollars for the costs of the two types of workers, but there is no clear explanation of how they arrived at that conclusion. At the top left of the poster, four rows of seven marks each are made to represent the artists; each group of seven is circled, showing that the cost of each of four artists is 7 dollars. To the right, there are three rows of three marks each, representing that each of three stonemasons earns three dollars each. There is no indication that any value other than the correct solution was considered and how. However, at the bottom of the poster the expressions written seem to indicate that an algebraic solution using simultaneous equations was employed.

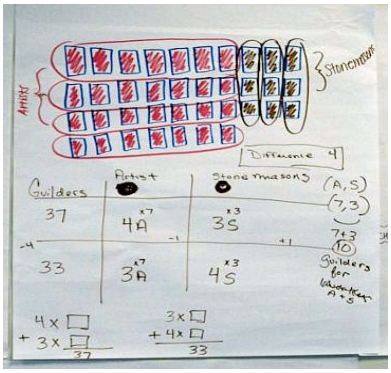

Figure 2. Tabular and pictorial

The third group, Figure 3, presented a *linear addition* solution using symbols to balance two equations. The top of the poster shows the two scenarios and the bottom of the poster shows the addition of these two. The top left of their poster shows three tens (squares) and seven units (circles) to represent thirty-seven dollars. On the right of the equal sign, there are four *A*s, for artists, and three *S*s, for stonemasons. Directly below that is a similar configuration to represent a thirty-three dollar cost for three artists and four stonemasons. The lower left portion of the poster shows six tens (squares) and ten units (circles) to represent seventy dollars. This is the addition of the tens (squares) and units (circles) from the two equations on the top of the poster. On the right side, there are seven *A*s and seven *S*s, which are the sum of the *A*s and *S*s from the two equations. This group reasoned that they now had a total of seven artists and seven stonemasons and a total of $70. They circled one artist, one stonemason, and the group of ten units (circles) to show that one of each worker would cost $10. This was one of the few groups who answered the question as written. They made no attempt to determine the individual costs for one artist or one stonemason. Members of this group were not unanimous about whether or not they should do so; however, several members of this group were confident that the question merely asked for the cost for one artist and one stonemason together and individual costs were not.

The fourth group, Figure 4, lists the two scenarios and then depicts the artists making 7 dollars each and the stonemasons making 3 dollars each. Below that, the group lists their check work. This seems to be *working backwards*. They reason that $33 and $37 added together equals $70; simultaneously, they reason that three artists and four stonemasons added to four artists and three stonemasons results in seven of each type of worker. If seven artists and seven stonemasons cost $70, the group reasons that one artist and one stonemason cost $10.

An interesting approach to finding the individual cost for each type of workers follows. First, the group realizes that both scenarios have three artists and three stonemasons. One scenario has an extra artist and the other scenario has an extra stonemason. Based on their conclusion that one artist and one stonemason cost $10, they derive that three artists and three stonemasons cost $30. Using this baseline, they argue that the scenario that has the extra artist is $37, which is $7 more than their baseline.

Therefore, the artist must cost $7. And, the scenario that has an extra stonemason costs $33, which is $3 more than their baseline. Therefore, the stonemason must cost $3.

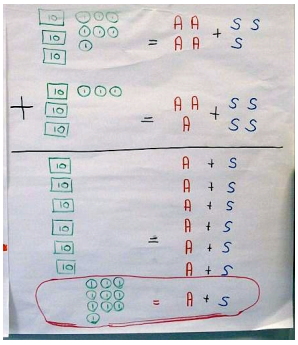

Figure 3: Linear addition

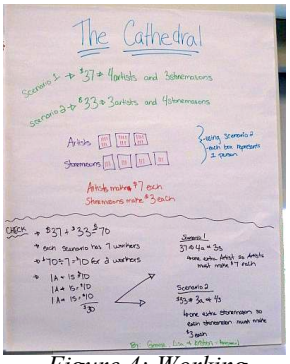

Figure 4: Working backwards

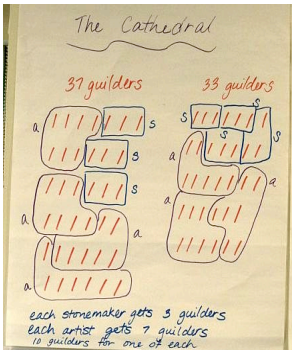

Figure 5: Working backwards

The fifth group, Figure 5, drew 37 hash marks on the left side of the paper and 33 hash marks on the right side of the paper. On the left side, there are three boxes drawn, each around three hash marks, and marked with an "s," for stonemason. And, the remaining hash marks are separated into four groups of seven by being encircled; each is marked with an "a," for artist. On the right side, there are four boxes drawn, each around three of the hash marks, and marked with an "s," for stonemason. The remaining hash marks are separated into three groups of seven by being encircled; each is marked with an "a," for artist. The poster, of course, only represents their final product and does not give insight to their thinking processes. While each poster seemed to represent a different thinking behind the solution strategy, teachers were able to make important connections between their respective poster proofs. All five of the possible representation (see figure 6) strategies were used by the groups: tables, pictures, graphs, numbers and symbols, and verbal descriptions.

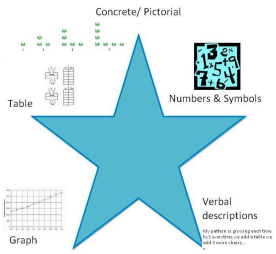

Figure 6: Multiple representations

A review of the artifacts from the other three groups (B, C, D) revealed a similar variety of approaches. Some groups used a strictly algebraic strategy with the simultaneous equations; however, there was a variety of other interesting approaches that were employed by the teachers in solving these equations beyond the traditional textbook approaches including substitution, linear addition or matrix approaches. One

group (Figure 7) started off with finding ways to arrive at partitioning the number 37, including 25 + 12, 26 + 11, and 27 + 10, even though none of these contain one value which is divisible by 7 and another value which is divisible by 3. Their last attempt, though, 28 + 9, does factor correctly. They then used the same strategy to arrive at partitioning 33, using 19 + 14, 20 + 13, and finally arriving at the correct 21 + 12.

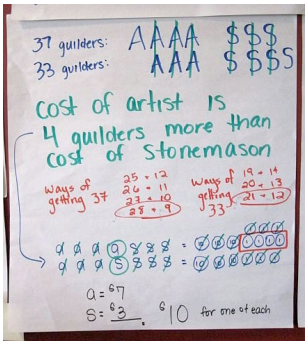

Figure 7: Working backwards

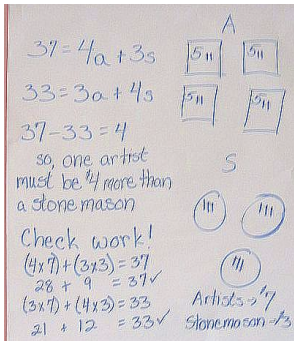

Figure 8: Working backwards

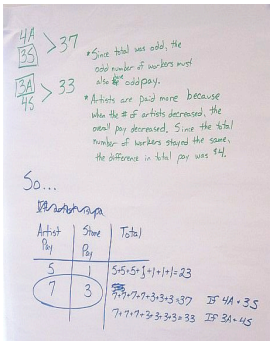

Figure 9: Working Backwards

Another group (Figure 8) first wanted to do a comparison to determine who made more. They reasoned that the cost with an extra artist was greater than the cost with an extra stonemason, concluding that artists cost more. Using algebra, they found that the cost for one artist was 4 dollars greater than the cost of a stonemason. They then drew four boxes to represent four artists and wrote a "5" inside each one. They also drew three circles to represent three stonemasons and drew one hash mark in each one, because stonemasons make 4 dollars less than artists do. They computed the total, 23 dollars and saw that it was short of the required 37 dollars. They added a hash mark to each square and circle and added the four 6s and three 2s to arrive at a total of 30, again too low. Adding one hash mark to each square and circle again, they added the four 7s and three 3s to get the required total of 37. From their picture, it was clear to see that an artist earns 7 dollars and a stonemason earns 3 dollars.

An interesting observation involving parity made by one of the groups (Figure 9) was that because the total cost in either scenario was odd and the number of total workers in each scenario was odd, then the individual pay for each type of worker must be odd. If there are four workers of the same type, then their total pay will be an even integer. However, the three remaining workers must have an odd wage, otherwise the total cost would be an even integer. Using the same logic in the second scenario shows that both types of workers must have an odd value for their daily pay.

Discussion and conclusion

As concluded by the teachers, conceptual understanding is the bedrock to advancement through any mathematics curriculum. The PD at the summer institute was centered on the deepening of understanding of rational numbers and proportional reasoning, the development of analytical skills and mathematical reasoning, and, a commitment to inquiry. The teachers themselves have voiced their beliefs that the struggles, the collaborations, and the iterative process have opened their eyes, allowing

them to discover how rational numbers behave; think about rational numbers and their properties in ways which they had not previously considered or experienced; and question their own beliefs about teaching proportional reasoning. The PD did not simply follow a script in autopilot. The discussions, which were such an integral part of the teachers' learning and discovery, were skillfully managed by the instructors. They were not mere bystanders; they engaged the teachers in productive conversations, listened to what the teachers were saying, and guided the discussions, as needed. They encouraged the teachers to share their ideas and kept the discussions focused on mathematical thinking, all while ensuring that professional courtesy was maintained. The teachers, in turn, will need to ensure that their classroom discussions also stay focused on mathematical thinking and approaches to proportional reasoning.

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Acknowledgement

The authors would like to acknowledge the Virginia Department of Education and the National Science Foundation for supporting this study.

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