The two-axis process model of understanding mathematics as a framework for designing mathematics lessons for high school students

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1. Introduction

The first author developed the 'two-axis process model' of understanding mathematics as a descriptive and prescriptive model in school mathematics (Koyama, 1993). As a result of theoretical and practical studies, he identified the principles and methods for designing school mathematics lessons based on the model (Koyama, 2005). Through a series of case studies in primary and lower secondary school mathematics, it has been demonstrated that the model can be used by teachers as an effective framework for designing mathematics lessons to improve students' mathematical understanding in a classroom (Koyama, 2010, 2013). However, we need more case studies on the students' process of understanding in upper secondary school mathematics. The purpose of this paper is to examine the effectiveness of the two-axis process model as a framework for designing mathematics lessons to improve students' understanding of mathematics in high school classrooms.

2. The two-axis process model of mathematical understanding

Based on theoretical studies that identify components for a descriptive and prescriptive model of mathematical understanding (van Hiele & van Hiele-Geldof, 1958; Wittmann,

Vertical axis (Levels of mathematical understanding)

- V1: Understanding of mathematical entities
- V2: Understanding of a relation of the entities
- V3: Understanding of the general relation

Horizontal axis (Stages of learning at each level)

- H1: Intuitive stage
  Students are provided opportunities for manipulating concrete objects, or operating on mathematical concepts and relations acquired in a previous level. At this stage, they do *intuitive thinking*.
- H2: Reflective stage
  Students are stimulated and encouraged to pay attention to their own manipulating or operating activities, to be aware of them and their consequences, and to represent them in terms of diagrams, figures or language. At this stage, they do *reflective thinking*.
- H3: Analytic stage
  Students elaborate their representations to be mathematical ones using mathematical terms, verify the consequences by means of other examples or cases, or analyze the relations among consequences in order to integrate them as a whole. At this stage, they do *analytic thinking*.

Figure 1. Two axes in the process model

1981; Pirie & Kieren, 1989), the first author made the two-axis process model of mathematical understanding (Koyama, 1993) as shown in Figure 1.

3. Designing mathematics lessons on the law of sines and the law of cosines for high school students in the 10th grade

We planned a series of research lessons on the law of sines and the law of cosines for high school students in a Japanese high school, and developed test items for the pre-test and the post-test to evaluate students' understanding of mathematics related to the law of sines and the law of cosines in the 10th grade.

3.1 Clarifying the levels of mathematical understanding

In the first step, we clarify the hierarchical levels of understanding mathematics contents in the 'Figure and Measurement' by analysing the intended curriculum (Ministry of Education, 2009), adopting the M1 method mentioned above. The two important viewpoints are identified. One is to understand the meanings of mathematical principles, concepts, and theorems. The other is to apply mathematical principles, concepts, and theorems in solving mathematical problems with a certain prospect. With these two viewpoints, we set up the levels of understanding of the law of sines and the law of cosines in the 'Figure and Measurement' in order to help students attain the level of understanding of the general relation as shown in Figure 2.

3.2 Embodying three stages of learning at each level

In the second step, we try to embody three learning stages (the above H1, H2, and H3) at each level to improve students' understanding of the law of sines and the law of cosines by adopting the methods M2 and M3 of the model mentioned above. We noticed that Japanese mathematics textbooks used in high schools have few mathematical problems to evaluate students' understanding of the generalizing process of mathematical theorems and formula in solving mathematical problems in the textbooks. On the basis of this textbook analysis and Figure 2, we embodied three learning stages at the level of understanding of a relation of the entities (V2) and the

<u>V1: Understanding of mathematical entities</u>

- To understand the meanings and definitions of three trigonometric ratios of an acute angel θ, their mutual relations, three trigonometric ratios of angle $180° - \theta$, and basic properties of geometrical figures learnt in primary and secondary school mathematics

<u>V2: Understanding of a relation of the entities</u>

- To make connections among the relations of side and angle in particular triangles by using trigonometric ratios and the Pythagorean theorem

<u>V3: Understanding of the general relation</u>

- To understand the meanings of the law of sines and the law of cosines
- To apply the law of sines and the law of cosines in solving mathematical problems with a certain prospect

Figure 2. Hierarchical levels of understanding of the law of sines and the law of cosines

<u>Three stages of learning at the V2 (understanding of a relation of the entities)</u>

H1: Intuitive stage

- To find out the length of side of in a particular given angle-side-angle triangle and a particular given side-angle-side triangle

H2: Reflective stage

- To reflect on trigonometric ratios and the Pythagorean theorem in the process of drawing an additional line in the given geometrical figure

H3: Analytic stage

- To think about the relation in other particular triangles

<u>Three stages of learning at the V3 (understanding of the general relation)</u>

H1: Intuitive stage

- To think about the relation found in the V2 in a general triangle

H2: Reflective stage

- To reflect on the investigation with particular triangles in the V2 to formulate the law of sines and the law of cosines

H3: Analytic stage

- To apply the law of sines and the law of cosines to particular triangles
- To understand the fact that the law of cosines is an extension of the Pythagorean theorem
- To apply the law of sines and the law of cosines in solving mathematical problems with a certain prospect

Figure 3. Three stages of learning of the law of sines and the law of cosines

level of understanding of the general relation (V3), respectively in the lessons of the law of sines and the law of cosines as shown in Figure 3. This is because the level of mathematical entities (V1) for understanding the law was supposed to have been attained by students to a certain degree as a result of learning mathematics in their previous grades. These stages were planned before conducting a series of mathematics lessons. Therefore contents of these learning stages could be changed and modified in the process of conducting lessons after due consideration of a result of the formative evaluation of students' understanding intended in the lessons.

4. A series of mathematics lessons and analyses of the data

On the basis of the clarified levels and the embodied three learning stages at each level, we made lesson plans of three 50-minute lessons for 40 high school students in the 10th grade. We also developed the pre-test and the post-test. In November 2012, the second author, a mathematics teacher, conducted the pre-test, research lessons, and the post-test in a Japanese high school. With permission from the school principal and the students to use the data collected during the lessons for our research, all tests were copied, and all lessons were video-recorded by his colleagues.

4.1 Quantitative analysis of the pre-test on mathematical entities for the lessons

Before starting the first lesson, the second author gave the pre-test to 38 students for twenty minutes to evaluate the students' understanding of mathematical entities related to the law of cosines. Results of the analysis showed that 33 students already attained at least the level of *certain understanding* of mathematical entities related to the law of cosines. Therefore we made a decision to start the mathematics lesson on the law of cosines according to the embodied stages of learning from the level of a relation of the entities up to the level of the general relation. Then we finalized the plans for two lessons on the law of cosines.

4.2 Mathematics lessons on the law of cosines and quantitative analysis of the post-test

The second author conducted two lessons on the law of cosines in one classroom with 40 students in the classroom as follows:

First lesson

In this lesson, the teacher first motivated the students' to find the length of the side of a particular given side-angle-side triangle, and asked the students to find the length of side in case of $\angle A = 60°$ by using what they had learnt. After solving a similar problem in case of $\angle A = 45°$, the students struggled to generalize the law of cosines of an acute triangle. This lesson did not include the law of cosines of an obtuse triangle because the plan was to give it to the students as a post-test item after two lessons.

Second lesson

In the next lesson, after reviewing the law of cosines of an acute triangle, the teacher gave the students two mathematical problems to be solved by using the law of cosines, and asked them to demonstrate their solutions on the blackboard. During the students' demonstration, the relation between the law of cosines and the Pythagorean theorem was confirmed. Additionally the students worked on the mathematical problem to find the length of side of a particular given side-side-side triangle.

Quantitative analysis of the post-test

At the end of the second lesson, the teacher gave the post-test (Figure 4) to 39 students for twenty minutes to evaluate the students' understanding of the law of cosines. We analyzed the students' responses to the post-test problems with the marking codes as follows:

Codes of Problem 1

(A) A student who draws the perpendicular line from the vertex B (or C) to AC (or AB), then uses the Pythagorean theorem
(A') A student who does same approach as a student of the code A, but does some calculation errors
(B) A student who uses the law of cosines itself
(Y) A student who gives wrong answer
(Z) A student who gives no answer

Codes of Problem 2

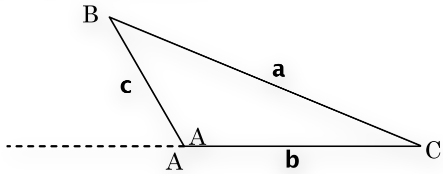
(A) A student who correctly prove the law of cosines in an obtuse triangle
(A') A student who draws the perpendicular line from the vertex B to AC, then express the length of a side, but does some calculation errors in using the Pythagorean theorem
(A'') A student who draws the perpendicular line from the vertex B to AC, then incompletely try to express the length of a side
(B) A student who draws the perpendicular line from the vertex A to BC
(Y) A student who gives wrong answer
(Z) A student who gives no answer

Problem 1 (Evaluation of the level V2)

Suppose $b = 6$, $c = 4\sqrt{2}$, and $A = 45°$ in a triangle ABC, find a by using the idea for getting the law of cosines.

Problem 2 (Evaluation of the level V3a)

Suppose the angle A is an obtuse angle in a triangle ABC, prove the law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$



Problem 3 (Evaluation of the level V3b)

Suppose $a = 8$, $c = 7$, and $B = 120°$ in a triangle ABC, find b by using the law of cosines.

Problem 4 (Evaluation of the level V3b)

Suppose $a = 4$, $b = 3$, and $c = 2$ in a triangle ABC, answer whether the angle A is an acute angle, right angle, or obtuse angle.

Figure 4. Post-test for evaluation of the levels V2 and V3

Codes of Problem 3

(A) A student who gives correct answer by using the law of cosines
(A') A student who correctly uses the law of cosines, but does a calculation error of $\cos 120°$
(A'') A student who correctly uses the law of cosines, but does some calculation errors
(Y) A student who gives wrong answer
(Z) A student who gives no answer

Codes of Problem 4

(A) A student who correctly uses the law of cosines to find $\cos A$, and answer that the angle A is obtuse
(A') A student who correctly uses the law of cosines to find $\cos A$, but does some calculation errors
(A'') A student who answer that the angle A is obtuse by comparing a^2 with $b^2 + c^2$
(Y) A student who gives wrong answer
(Z) A student who gives no answer

Table 1 shows the results of the post-test of 39 students. We notice that there is the vivid difference between the response patterns in Problem 2 and Problems 3 and 4. It is important for us to remember the fact that Problem 2 is intended to evaluate students' understanding of the meanings of the law of cosines while Problem 3 and Problem 4 are intended to evaluate their ability in applying the law of cosines in solving mathematical problems with a certain prospect. As a result of analyzing all the problems in detail, all students who gave the correct answer to Problem 2 (Code A) can correctly answer

Problem 3 and Problem 4 (Code A or A'). Additionally these students attained at least the level of *certain understanding* of mathematical entities related to the law of cosines in the pre-test. On the other hand, 8 of 11 students who drew the perpendicular line from vertex A to BC in Problem 2 (Code B) used the law of cosines in Problem 1 (Code B).

Table 1. Result of the post-test for evaluation of the levels V2 and V3 (39 students)

Code	Problem 1	Problem 2	Problem 3	Problem 4
A	20	11	33	29
A'	1	3	2	2
A''	---	10	2	4
B	18	11	---	---
Y	0	1	2	3
Z	0	3	0	1

4.3 Mathematics lesson on the law of sines and qualitative analysis of the lesson

In rethinking our plan for the third lesson on the law of sines, we focused on the fact that only 7 of 20 students who drew the perpendicular line from vertex B (or C) to AC (or AB) then used the Pythagorean theorem (Code A) in Problem 1 correctly proved the law of cosines in an obtuse triangle (Code A) in Problem 2. It means that the students' understanding of mathematical entities related to the law of cosines does not always guarantee the improvement of their understanding up to the higher level of understanding of the general relation in the two-axis process model. We made a careful reflection on the problems given to the students in the first lesson on the law of cosines. As a result of our reflection, we thought that the two problems that asked for the length of side in particular side-angle-side triangles in cases of $\angle A = 60°$ and $\angle A = 45°$ were not effective to shift the students' awareness to the law of cosines. The reason is that students could solve these problems without using any trigonometric ratios. Therefore, there could be a gap between the solutions to these problems and the generalization of the law of cosines of the general triangle.

Third lesson

Based on this formative evaluation of the first lesson, we decided to replace one problem with another that could not be solved by only using the Pythagorean theorem in the third lesson on the law of sines. We then planned the third lesson by embodying three stages of learning of the law of sines with the following two mathematical problems, Problem 1 and Problem 2 that asked for the length of side in a given angle-side-angle triangle, and one key question to help the students shift their awareness from the particular to the general law of sines.

Problem 1: Suppose $a = \sqrt{6}$, $\angle B = 45°$, and $\angle C = 75°$ in a triangle ABC, find b.

Problem 2: Suppose $a = 4$, $\angle B = 55°$, and $\angle C = 82°$ in a triangle ABC, find b.

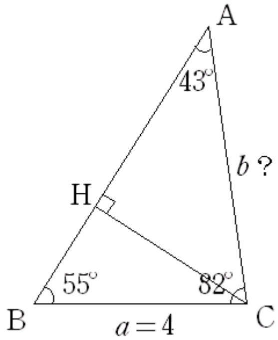
Key question: If a, $\angle B$, and $\angle C$ are given in a triangle ABC, can you find b?

Qualitative analysis of the lesson on the law of sines

In the process of their generalizing the law of sines, the students reflected on one peer student S3's solution (Figure 5) to Problem 2 in a classroom discussion. The analysis of the videotaped dialog between the teacher and another student S2 who found a gap between the solution to Problem 2 in the post-test (Figure 4) and the generalization of the law of cosines confirmed that Problem 2 in the third lesson and the students' discussion of its solution were effective in promoting the students' reflective thinking about the relation of side and angle (i.e. a relation of mathematical entities) leading to the law of sines (i.e. the general relation) in the classroom.

Problem 2 in the third lesson on the sine rule

Suppose $a = 4$, $\angle B = 55°$, and $\angle C = 82°$ in a triangle ABC, find b.



$$CH = a\sin 55° = 4\sin 55°$$

$$\begin{aligned} b = AC = CH\frac{1}{\sin 43°} \\ &= \frac{4\sin 55°}{\sin 43°} \\ &= \frac{4 \times 0.8192}{0.6820} \\ &= \frac{8192}{1705} \\ &= 4.8 \end{aligned}$$

Figure 5. A student S3's solution to Problem 2 on the blackboard

5. Conclusion

As a result of the quantitative and qualitative analyses of the collected data during the pre-test, research lessons, and the post-test the following two points emerged. Firstly, about the level of understanding of general mathematical relations, all students who understood the meanings of the law of sines and the law of cosines and how to prove these laws can use them in solving mathematical problems, while the opposite is not true. Secondly, in order to shift the level of their understanding from the level of mathematical relation of the entities up to the level of general mathematical relations, students beforehand need to understand the law of sines and the law of cosines as the relations of mathematical entities with particular examples in a certain degree. These two findings verify the effectiveness of the two-axis process model as a framework for designing mathematics lessons to develop students' understanding of mathematics in high school classrooms.

The first finding suggests that when a teacher designs mathematics lessons for teaching and learning in the teaching unit of the 'Figure and Measurement,' it is useful for the teacher to make a distinction between the understanding of the meanings of the law of sines and the law of cosines and in using these laws to solve mathematical problems at the level of general mathematical relations. This distinction is needed for a teacher to evaluate students' understanding of the general mathematical relations, and to clarify the levels of mathematical understanding in the two-axis process model. The second finding suggests that it is important for a teacher to aim for students' understanding of the relations of mathematical entities to shift their understanding up to the level of general mathematical relations according to the two-axis process model. In the future research, we need more case studies on embodying three learning stages with possible students' mathematical activities and a teacher's roles in mathematics lessons in order to improve students' understanding of mathematics in high school classrooms.

Endnotes

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