Instruction on the center of gravity

Masahiro Takizawa, Kuroiso Senior High School, Japan

Introduction

In Japan, students learn about the center of gravity both in mathematics and physics. However, the meaning of the center of gravity is different in each. In physics, it means the balance point of a figure, while in mathematics it means the intersection of the median lines of a triangle. In a triangle, both points are the same. In textbooks, there are no explanations why both points are the same. The author designed a plan and material, which joins the two different meanings. In the material, generalization, specializing, and clarification of the definition of the center of gravity are repeatedly used and these questions can be answered. The author tried the material as a classroom practice. The students were impressed by the mathematical thinking and had valuable experiences of the modeling process.

Modeling process

Miwa (1983) illustrated the modeling process as shown in Fig. 1 and wrote that this process typically consists of the following four steps;

(1) to formulate the situation into a problem in mathematical terms (Formulation),
(2) to determine mathematical results (Mathematical work),
(3) to interpret and evaluate the results with respect to the original situation (Interpretation and evaluation),
(4) to improve the model to get better results (Improvement of model)

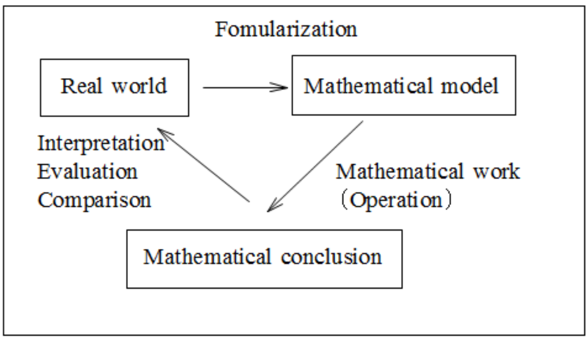

Figure 1. Modeling process

The author proposed some models from this point of view (Takizawa, M. 1998b, 2001, 2012). The students were impressed by mathematical thinking and such models are useful and should be included in the curriculum. However, these materials are lacking at the time of writing of (4). In this paper, the material includes all the points, as noted later on.

Ikeda (1999) wrote that following two assimilated ways of thinking should be expected in mathematical modeling:

(1) Justifying conditions or assumptions, which we simplify or idealize;
(2) Starting from the simple model and getting closer to the real events gradually.

Ikeda illustrated the inner process of thinking as shown in Fig. 2 and the model should involve the interactive action of the figure shown to improve the model to get better results.

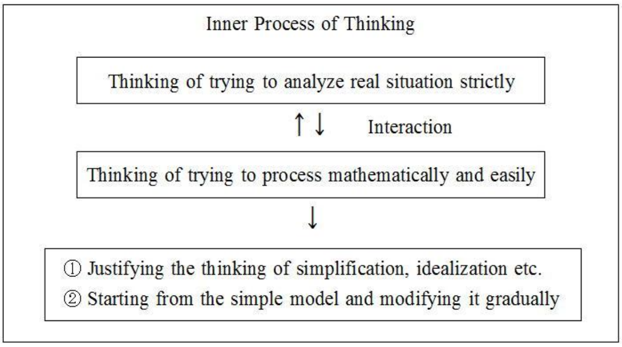

Figure 2. Interaction to improve model to get better results

In other words, there are three roles to be considered in designing a model:

One person to affirm the model because of easy processing;

One person to negate the model because of differences between the model and real situation;

One person to consider in a comprehensive way.

The following materials give many examples of interaction with the students in searching for the point of center of gravity. Finally, the model is modified in a satisfactory manner.

Plan and material

Moment of force

Generally speaking, balanced right and left arms of scales mean that the moment of force are the same in both arms. For example, assume each weight of counterweight is equal to 1 and the distances from the fulcrum are as shown in figure 3. Then, the left moment is $4\times1+3\times1=7$ and the right moment is $2\times2+3\times1=7$. So, the two sides are balanced.

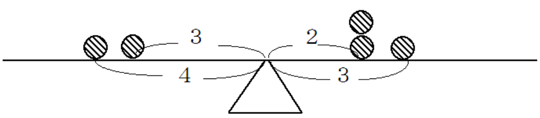

Figure3. A balanced arm

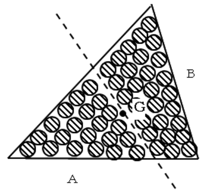

Figure 4. A triangle with full points is balanced on the axis

Figure 4 shows a triangle separated into 2 parts by a straight line passing through point G. Then, if point G is the center of gravity, the moments of the points on both sides are the same for <u>any</u> straight line. Is there such a point in the triangle?

The author designed the visual basic (VB) program named "Balance" (Fig. 5) that compares the arms of scales to a number line and compares the fulcrum to the origin. There are many "+" and "-" buttons on the screen. Clicking any "+" button increases the number of balls. Clicking any "-" button decreases the number of balls. Comparing the number line to the arm allows us to use the following statement:

If the coordinate of counterweight i is x_i and the weight is $m_i\ (i = 1 \cdots n)$ then when the value of $m_1 x_1 + m_2 x_2 + \cdots + m_n x_n$ is 0, the arm is balanced.

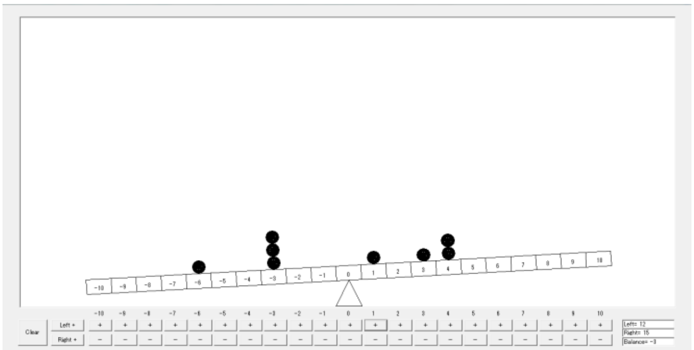

Figure 5. The arm is not balanced if moments on both sides are different.

Figure 6 shows the triangle on the number line and we can see that the right and left arms are balanced. We can confirm that the center of gravity is on the y axis and BO:CO=1:2. Figure 7 shows that the average of x-coordinates of the vertices is 0. Using "Balance," students can try various kinds of triangles and confirm that point G (intersection of the median lines) is the center of gravity.

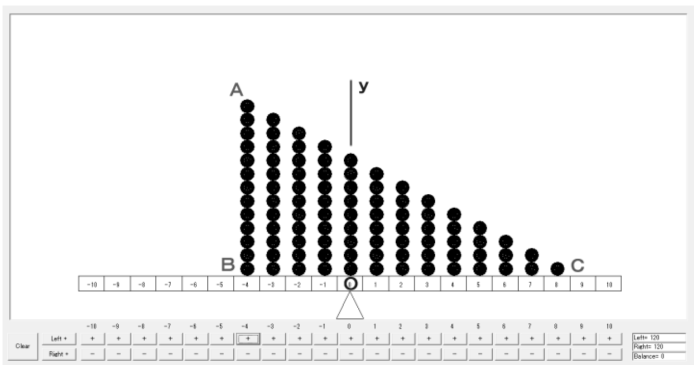

Figure 6. The right triangle is balanced. BO:CO =1:2

Seeking point G using an integral

In fact, there are infinite points on the triangle. However, we can confirm only the case of finite points using "Balance."

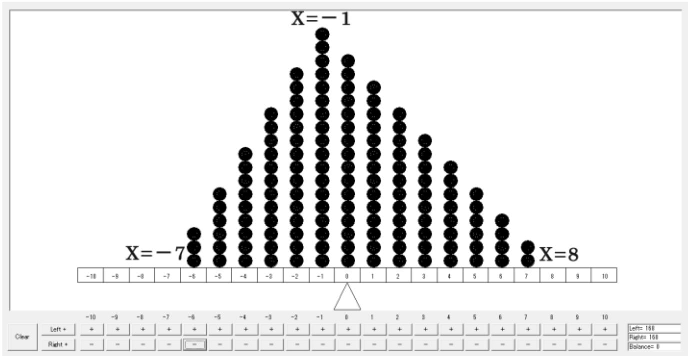

Figure 7. The average of the x coordinates is 0.

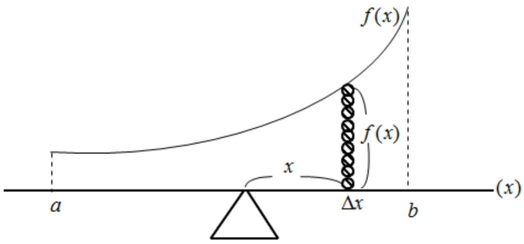

Figure 8. Seeking the center of gravity in a figure with infinite points

Assume that the place of fulcrum is 0 and the height of the figure with x distant from the fulcrum is $f(x)$ (Fig. 8). Then, the mass of this point is $f(x)\Delta x$ and the moment of force is $xf(x)\Delta x$. So, if the figure is balanced, then $\sum_i x_i f(x_i)\Delta x = 0$.

When $\Delta x \to 0$, the upper expression becomes $\int_a^b xf(x)dx = 0$. For example, in the case of $y = 4 - x$ $(-2 \le x \le 4)$ (Fig. 9), center of gravity must be 0. We can confirm that the center of gravity is on $x = 0$.

$$\int_{-2}^{4} x(4-x)dx = \int_{-2}^{4}(4x - x^2)dx = \left[2x^2 - \frac{1}{3}x^3\right]_{-2}^{4} = 0$$

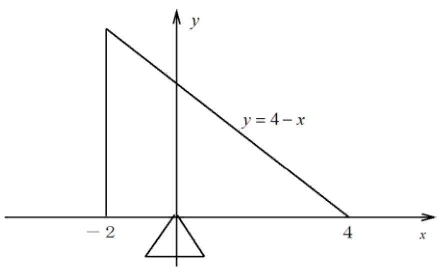

Figure 9. The center of gravity is on the y axis.

However it cannot be a general proof. We have to confirm in the case of a triangle of which vertices are $A(a,b)$ $B(c,d)$ $C(-a-c,-b-d)$. It is very complex, requiring lengthy calculations and therefore it will be difficult for senior high school students.

Center of gravity of a quadrangle

We can easily guess that the coordinates of the center of gravity for a quadrangle with vertices $A(x_1,y_1)\ B(x_2,y_2)\ C(x_3,y_3)\ D(x_4,y_4)$ are

$$G\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4}\right)$$

The sum of the x coordinates is set to be 0 in the following quadrangle. Confirm the center of gravity using integral calculation (Fig. 10).

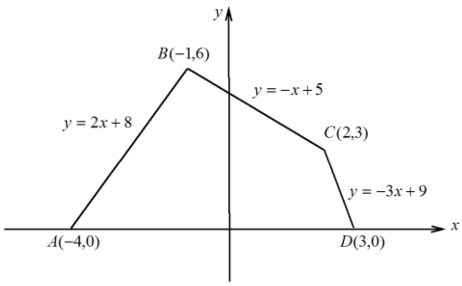

Figure 10. A quadrangle having an x coordinate average=0

$$\int_{-4}^{-1} x(2x+8)dx+\int_{-1}^{2} x(-x+5)dx+\int_{2}^{3} x(-3x+9)dx=-10$$

If it were balanced, the integral should be equal to 0. The value shows that the center of gravity does not exist on the y axis.

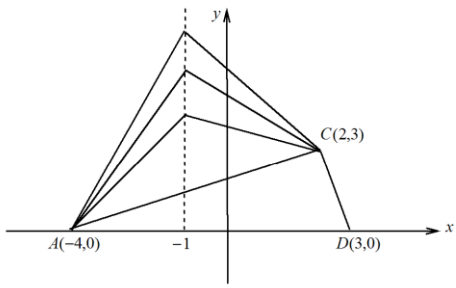

Figure 11. Multiple quadrangles having an x coordinate average=0

If you think about it, there are many quadrangles of which the sum of the x coordinates is 0 (Fig 11). It is clear that each moment is different. This is the evidence that G is not a center of gravity. Then what is the point $G\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4}\right)$ for a quadrangle?

To answer this question, assume a figure consisting of only 4 points with coordinates $A(x_1,y_1)\ B(x_2,y_2)\ C(x_3,y_3)\ D(x_4,y_4)$. Fig. 12 is an example.

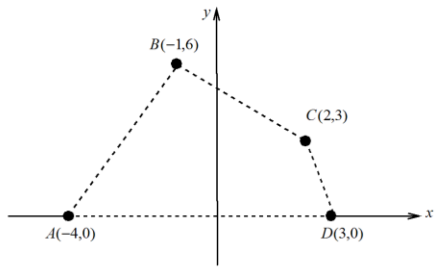

Figure 12. A figure consisting of 4 points

To make it less complex, focus only on the x coordinate. Assume there are only four counterweights on the arm whose coordinates are x_1, x_2, x_3, x_4 and the coordinate of the balanced point is x (Fig. 13), then the moment from x to x_i is $(x_i - x)\cdot 1 = x_i - x$. As the sum of moment equals 0, $\sum_{i=1}^{4}(x_i - x) = 0$, $x = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{4}$.

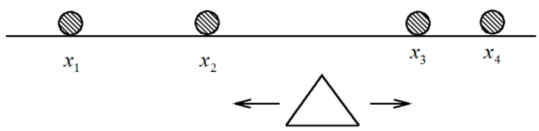

Figure 13. Finding the center of gravity

This means that the average point is the center of gravity. So, this figure is balanced when the fulcrum is located on the axis point $x = \frac{x_1 + x_2 + x_3 + x_4}{4}$. Similarly, it is balanced when the fulcrum is on the axis point $y = \frac{y_1 + y_2 + y_3 + y_4}{4}$. So, $G\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}\right)$ is the center of gravity in the figure, which consists of only four points. Similarly, in the case of a triangle consisting of only three points with coordinates $A(x_1, y_1)\, B(x_2, y_2)\, C(x_3, y_3)$, the center of gravity is $G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$. So, in the case of a triangle, $G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$ is a center of gravity both in the case of only three points and when the contents of the figure are full points. However in the case of a quadrangle, we have to distinguish whether the contents are full or not.

Is $G\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}\right)$ *really a center of gravity?*

We have doubts about the balance point of a quadrangle consisting of four points. The condition that the point G is a center of gravity is that the figure has to be balanced on <u>any axis</u> going through the point G. We can only confirm in the case of two axes. (Fig. 14). The proof must be for any axis going through the point G; Vectors are very useful for this proof.

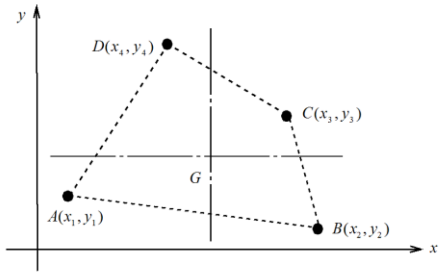

Figure 14. A quadrangle balanced on two axes.

First, point G is be moved to the origin O (Fig. 15). The vector in the horizontal direction from the y axis to each point set becomes $\vec{e}_i\ (i=1,2,3,4)$. As the figure is balanced in the y axis, $\vec{e}_1+\vec{e}_2+\vec{e}_3+\vec{e}_4=\vec{0}$.

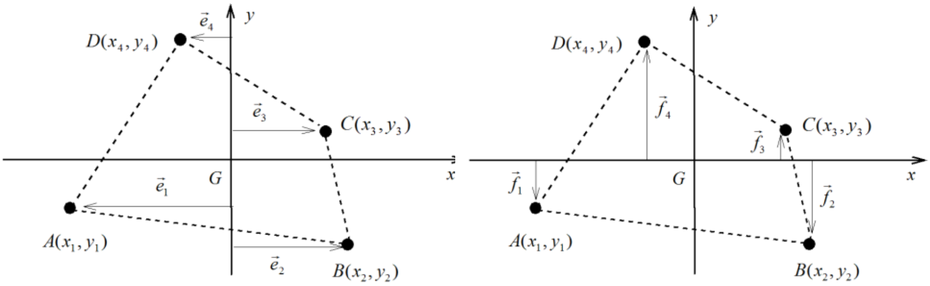

Figure 15. The vector sum in these quadrangles= $\vec{0}$

Similarly, the vector in the vertical direction from the x axis to each point is set to $\vec{f}_i\ (i=1,2,3,4)$, then $\vec{f}_1+\vec{f}_2+\vec{f}_3+\vec{f}_4=\vec{0}$. As the vector from point G to each point is $\vec{e}_i+\vec{f}_i\ (i=1,2,3,4)$, then $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}+\overrightarrow{GD}=\vec{0}$. Next, pivot this figure round the point G. (Fig. 16 (Left)). After turning, the expression $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}+\overrightarrow{GD}$ changes into $\overrightarrow{GA'}+\overrightarrow{GB'}+\overrightarrow{GC'}+\overrightarrow{GD'}$ and $\vec{e}_1+\vec{e}_2+\vec{e}_3+\vec{e}_4$ changes into $\vec{e'}_1+\vec{e'}_2+\vec{e'}_3+\vec{e'}_4$ (Fig. 16 (Right)). It is clear that $\overrightarrow{GA'}+\overrightarrow{GB'}+\overrightarrow{GC'}+\overrightarrow{GD'}=\vec{0}$ and $\vec{e'}_1+\vec{e'}_2+\vec{e'}_3+\vec{e'}_4=\vec{0}$.

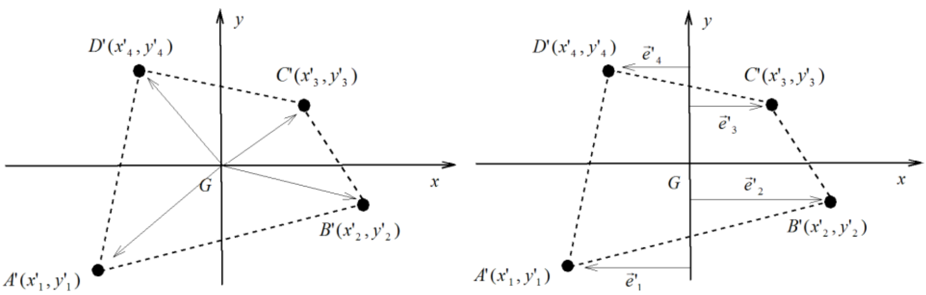

Figure 16. (Left) The quadrangle after rotation (The rotation axis passes through the point G) (Right) Then the sum of these vectors is $\vec{0}$

Therefore, for any axis going through the point G, the figure is balanced. This is the true proof that the point G is the center of gravity of the figure.

Finding a center of gravity using orthogonal coordinate system

We think of the center of gravity using the orthogonal coordinate system. However we can do the same thing using the oblique coordinate system (Fig. 17).

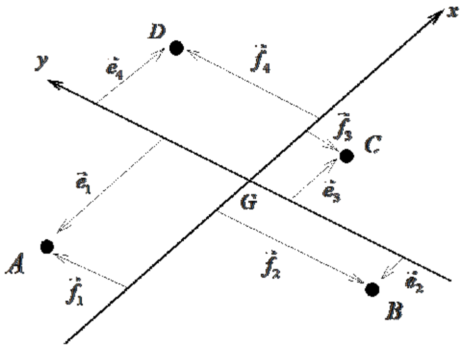

Figure 17. Vectors in an orthogonal coordinate system

That is, if $\vec{e}_1+\vec{e}_2+\vec{e}_3+\vec{e}_4=\vec{0}$ and $\vec{f}_1+\vec{f}_2+\vec{f}_3+\vec{f}_4=\vec{0}$, then $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}+\overrightarrow{GD}=\vec{0}$ and point G is the center of gravity of the figure. In other words, there are two oblique axes and if each of the two sums of the vectors parallel to the oblique axis is $\vec{0}$, then the intersection of the oblique axes is the center of gravity of the figure (Fig. 18). The number of the points is not limited to four.

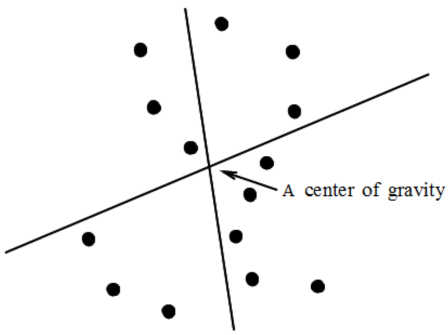

Figure 18. The intersection of the oblique axes is the center of gravity.

Reconsideration for a center of gravity of a triangle with full points

In Figure 19, CM is a median line. Draw a segment parallel to the line AB in such a way that the intersection of CM with the segment becomes a middle point. If the number of the points on the segment were finite, the points on the segment are separated in half by the median line. So the sum of the vectors from the median line to each point on the segment is the null vector $\vec{0}$. That is, the total sum of the vectors in Figure 19 becomes the $\vec{0}$. This means that the center of gravity of a triangle is on the median line.

Therefore, the center of gravity is the intersection of the median line. If we increase the number of points to infinite, the place of the balance point does not change.

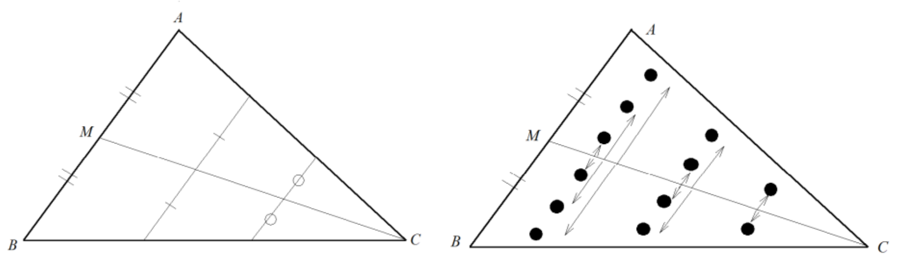

Figure 19. The points in a triangle are divided in two making the sum of the vectors to become the null vector $\vec{0}$.

Reconsideration for a balance point of a quadrangle

We can now calculate the balance point of a quadrangle, which consists of full points. Separate the quadrangle into two parts, ΔABC and ΔACD . Assume G_1 is the center of gravity ΔABC and G_2 is center of gravity of ΔACD (Fig 20). Moreover assume points $P_1, P_2, \cdots P_m$ are included in ΔABC and points $Q_1, Q_2, \cdots Q_n$ are included in ΔACD .

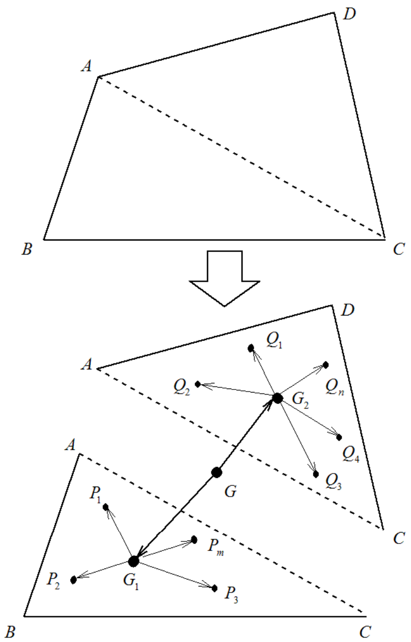

Figure 20. The quadrangle is divided into two triangles making point G to be located between G_1 and G_2

The following expressions hold.

$$\overrightarrow{G_1P_1}+\overrightarrow{G_1P_2}+\overrightarrow{G_1P_3}+\cdots+\overrightarrow{G_1P_m}=\vec{0} \qquad \overrightarrow{G_2Q_1}+\overrightarrow{G_2Q_2}+\overrightarrow{G_2Q_3}+\cdots+\overrightarrow{G_2Q_n}=\vec{0}.$$

If G is the center of gravity of the quadrangle, the following expression holds.

$$\overrightarrow{GP_1}+\overrightarrow{GP_2}+\cdots+\overrightarrow{GP_m} \quad + \quad \overrightarrow{GQ_1}+\overrightarrow{GQ_2}+\cdots+\overrightarrow{GQ_n}=\vec{0}.$$

The left side of this expression is transformed into the following expression.

$$\begin{aligned}
&\overrightarrow{GG_1}+\overrightarrow{G_1P_1}+\overrightarrow{GG_1}+\overrightarrow{G_1P_2}+\cdots+\overrightarrow{GG_1}+\overrightarrow{G_1P_m}\\
&+\overrightarrow{GG_2}+\overrightarrow{G_2Q_1}+\overrightarrow{GG_2}+\overrightarrow{G_2Q_2}+\cdots+\overrightarrow{GG_2}+\overrightarrow{G_2Q_m}\\
&=m\overrightarrow{GG_1}+n\overrightarrow{GG_2}+\overrightarrow{G_1P_1}+\overrightarrow{G_1P_2}+\cdots+\overrightarrow{G_1P_m}+\overrightarrow{G_2Q_1}+\overrightarrow{G_2Q_2}+\cdots+\overrightarrow{G_2Q_n}\\
&=m\overrightarrow{GG_1}+n\overrightarrow{GG_2}
\end{aligned}$$

Therefore $m\overrightarrow{GG_1}+n\overrightarrow{GG_2}=\vec{0}$.

Then G is a point divided internally between G_1 and G_2. The rate is n:m. And n:m means the rate of areas ΔACD and ΔABC. The center of gravity G is a point located internally between G_1 and G_2, $\Delta ACD : \Delta ABC$.

Results and discussion

Classroom practice & students' impressions

The author taught the center of gravity of figures using the "Balance" program to eleventh-grade students. They showed great interest in making efforts to put the points in order to make a balanced figure of which the fulcrum is the origin. They tried various kinds of figures as well as triangles and found the following thing by themselves: If the coordinate of each counterweight i is x_i then the figure is balanced if and only if $x_1+x_2+\cdots+x_n=0$. They were able to reach the concept of moment easily and found that average point is the center of gravity.

Figure 21. Students in the lesson

The author then changed the subject to infinite points. However nothing resulted from them. So, the author instructed the integral calculation which the students found monotonous. However, when they were able to confirm that the center of gravity is different from the average point, some students had conflicting ideas because of their contradicting expectations.

The author got back to the subject of finite points. Some students doubted whether $G\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}\right)$ is a center of gravity of four points.

"The figure has to be balanced on any axis going through the point G!" a student said. However no one was able to confirm it using any mathematical method. The author then proved it using vectors. The students were able to confirm that $G\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4}\right)$ is really a center of gravity. Some students were still not convinced and they finally came to the conclusion that they have to distinguish between finite points and infinite points. Since there was no more time to consider the rest themselves, the author taught the remaining concepts to the students. They clearly came to see that the intersection of medians is the center of gravity by using vectors.

Many different impressions were expressed. Some impressions are as follows:

"For me, practical trial to make a balanced figure was a very wonderful experience. It is easy to understand the center of gravity using the program."

"Mathematics is deep and interesting! I was able to extend my world by learning the place of the center of gravity."

"I got to know that there is a law and regularity for balanced figures. This program is fantastic and convenient to find a place of center of gravity."

"By thinking of average, I was able to arrange the points on the screen and make a balanced triangle."

"I think that vectors are very applicable to prove the nature of figures. I want to use vectors to apply mathematics to something. Vectors are a good method of thinking!"

"It was my fun to put the points in order to make a balanced figure. I realized that there is a connection between real figures and mathematical theory. It was my first experience to find the applied aspect of mathematics."

As we can see from the students' impressions, seeking the place of center of gravity of figures make the students gain interest and help them understand the meaning about center of gravity deeply. From these student's impressions, it can be seen that they gained motivation in the study of difficult mathematical concepts that hopefully will stimulate them in trying to understand other difficult concepts not only in mathematics. If the students had enough time in the lesson, they could expand the lesson by cutting paper figures using scissors, attaching a thread to the figure, and confirming the location of the center of the gravity based on the calculations. This process translates from the mathematical conclusion to the real world.

Koyama (2012) did the practical study for the Ceva's and Menelaus' Theorem using the concept of the center of the gravity. In the lesson the students found the center of the gravity of a triangle by hanging different weights on its vertices and using a balance. His experimental study is highly regarded. Miwa (1983) identified the modeling process consisting of the following processes in the case of triangle or quadrangle with full points in the real world: experimental verification using the program, verification using integral calculation, simplification to finite points of a figure, distinction for the definition of the center of the gravity between finite and infinite points of a figure, and proofs for the hypothesis by thinking of vectors and generalization or expansion to infinite points of figures. The process includes the improvement of the model to get better results. It was inevitable when they expanded finite points to infinite points. Many students thought that the theory of finite points lent itself well to the case of infinite points. However, integral calculation denied this hypothesis. They had to analyze the real situation again. As a consequence, the model itself changed into a more practical model based on reality.

As Ikeda (1999) wrote, there were interactive actions between justifying the thinking of simplification or idealization and starting from the simple model and modifying it gradually. In this model, the former is simplifying to finite points of figures and seeking the location of the center of gravity and the latter is expanding to infinite points of figures. The students were able to experience these two ways of thinking. It is interesting that some students had conflicting ideas in these interactive actions.

References

Ikeda, T. (1999). A study of thinking to facilitate Mathematical modeling, *Journal of Japan Society of Mathematical Education Research in Mathematical Education Vol. 71,72* , 3-18 (in Japanese)

Koyama, N (2012). Initiation of Ceva's and Menelaus' Theorem with center of gravity, *Journal of Japan Society of Mathematical Education Volume XCIV* No.3, 9-16 (in Japanese)

Takizawa, M. (1998b). Maps and Mathematics, *Journal of Japan Society of Mathematical Education Volume LXXX* No.11, 17-22 (in Japanese)

Takizawa, M. (2001). Calculating paths in a map using a matrix, *Proceedings of the Sixth Asian Technology Conference in Mathematics, ATCM, Inc*, 333-342,

Takizawa, M. (2012). Colors and mathematics, *Pre-proceedings of the 12th International Congress on Mathematical Education (Topic Study Group 17) ,COEX, Seoul, Korea 3431-3441*

Miwa, T. (1983). Study on the modeling in Mathematics Education, *Tsukuba Journal of Educational Study in Mathematics, Mathematics Education Division, Institute of Education, University of Tsukuba*, 117-125 (in Japanese)

Masahiro Takizawa
Kuroiso Senior High School
Yutakamachi 6-1, Nasushiobara City, Tochigi Prefecture,
Japan
Takichan_family_webclub@yahoo.co.jp