Pupils' interpretations of a problem situation: The case of population density

Mai Hirabayashi, University of Tsukuba, Japan

Introduction

Mathematical modeling is a means for interpreting problems in real life mathematically. It is composed of the following activities: one makes real models from original situations, makes mathematical models from the real models, obtains mathematical results from the mathematical models, and leads to conclusions from the results (Blum, 1993). Previous studies on modeling have focused mainly on educational stages beyond the middle school (e.g., Niss, 2010). However, pupils in elementary school may also be able to interpret a problem situation mathematically to some extent. Hence, there is a need to examine pupils' behaviors in modeling in order to modeling in elementary school.

The purpose of this study is to explore how pupils interpret the situation in a modeling activity. To achieve this purpose, the current study developed a set of problems, including one with population density, and analyzed pupils' responses to them.

Interpretations of a problem situation in mathematical modeling

Mathematical modeling process

Lesh and Zawojewski (2007) proposed the following definition: "a task becomes a problem when the problem solver needs to develop a more productive way of thinking about the given situation" (p. 782). In order to "develop a more productive way of thinking," the solver needs to engage in a process of interpreting the situation, which in mathematics means modeling. Thus, problem solving from Models-and-Modeling Perspective (MMP) is defined as the process of interpreting the situation mathematically. This process involves several iterative cycles of expressing, testing and revising mathematical interpretations, and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics (ibid.). This study is based on the framework proposed by Lesh and Zawojewski (2007), that is, mathematical modeling is a process of interpreting a situation mathematically.

The theoretical framework for describing qualitative differences of models

In problem solving from MMP, children's activities that bring their own personal meaning to bear on a problem, test and revise their interpretations when model-eliciting activities are given. In the activities, children's models are complex artifacts that need to be useful for a given client in a given situation and those artifacts need to be sharable and reusable in other situations, for other data sets, or by other people. Hence, children need to go through cycles of testing and revising models (Lesh and Zawojewski, 2007, pp. 783-784).

Children's trial models tend to be expressed using a variety of representational media, including spoken language, written symbols, graphs and graphics, formulae and experience-based metaphors (Lesh and Zawojewski, 2007, p. 785). That is, there are two types of models, one expressed by non-mathematical means and the other by mathematical means. In this study, the former is called "pre-mathematical model" and the latter is called 'mathematical model,' which is written symbols, graphs and graphics, formulae and so on.

It isn't necessary that children can judge and use adequate mathematical means. At first, they may use models based on looking and feeling to interpret the situation. These are pre-mathematical models. Then, models seem to develop from pre-mathematical models to mathematical models by testing and revising these models.

For example, models for the Big Foot Problem (illustrated in Lesh & Doerr, 2003) are exemplified. The problem is to make a how-to-toolkit that police can use to make good guesses about how big people are – just by looking at their footprints. One makes models, which may make use of qualitative judgments about the size of footprints of people of different sizes and gender or from wearing different types of shoes (e.g., *This guy's huge. The tread's just like mine.*). Then, one expresses the size of the feet and height mathematically. For example, one is thinking in terms of additive differences as "if one footprint is 15 cm longer than another, then the difference in height is also 15 cm." Thus, the models develop from a pre-mathematical to a mathematical model.

The mathematical models in themselves seem to develop depending on the qualitative difference of a structural feature. For example, the additive model mentioned above is useful for a particular person, but not for another. Thus, it is necessary to revise the model to be used in other situations. Based on this need, one revises the additive model into a proportional model as "a person's height is estimated to be about six times the size of the person's footprint." The qualitative difference of a mathematical model is set depending on the structure of the problem situation.

The qualitative difference of models is summarized in Table 1. It isn't necessarily implied that a pre-mathematical model is a starting point and models develop in that order.

Table 1. The qualitative difference of models

<table>
<tr><th>Level</th><th>Models</th></tr>
<tr><td>1</td><td>Pre-Mathematical Model</td></tr>
<tr><td>2</td><td rowspan="3">Mathematical Model</td></tr>
<tr><td>⋮</td></tr>
<tr><td>n</td></tr>
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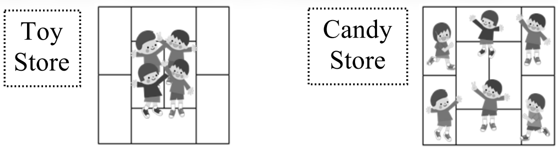
The model is elicited as a result of interpreting a problem situation. If the model develops, then the interpretations of the situation in itself also develop. "Understanding of the situation is not thought of as being all-or-nothing situation" (Lesh & Zawojewski, 2007, p. 783). Therefore, it is important that the researchers and the teachers focus their attention on the qualitative difference of models.

Materials and method

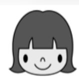
The test

The current study developed a set of problems in order to explore how pupils interpret a problem situation. This paper shows the one with population density (see *Figure 1*).

Mai went to a toy store and a candy store. There are four children in the toy store that is an eight-mat room. There are six children in the candy store that is an eight-mat room.



Mai is thinking which room is crowded.



Four children are in the middle of the room. So I think that the toy store is crowded.

Figure 1. The problem of population density

Generally speaking, the population density is expressed by a ratio that compares different kinds of quantities. On the other hand, we feel crowded when it is more crowded with people despite the extent of space in real life, as in a jam-packed train, a queue and so on. The pupils may also feel these "daily population density" intuitively. In the test, the figure, which shows a situation that four children are in the middle of a room, is given in order to bring out pupils' pre-mathematical model. Moreover, the area of both stores is made the same. The pupils may focus on one quantity or two different kinds of quantities. These are qualitative differences in the mathematical models themselves. That is, the area of both stores is made the same in order to bring out qualitative differences of pupils' mathematical models.

Procedure and participants

The participants were 327 pupils from third grade (9-year olds) to sixth grade (12-year olds) in Japan. The pupils of fifth and sixth grade have already learned the ratio of a quantity per unit. But the pupils of third and fourth grade have not learnt that. The author wanted to study if they could interpret a problem situation by using the similar idea.

Table 2. The subjects of this study

Grade	The number of pupils
Third	98
Fourth	66
Fifth	99
Sixth	64
Total	327

Data analysis

It is appropriate to express the population density as a ratio that compares different kinds of quantities, so Mai's idea isn't correct. But because of the purpose of this study, the qualitative difference in the pupils' interpretations is analyzed. In this paper, the pupils' descriptions representing the population density are interpreted as the models made by the pupils.

Results

The results of pupils' response are displayed in Table 3. Most of the pupils opposed Mai's idea, and there is not much difference between each grade.

Table 3. The proportion (in percentages) of pupils' response

Grade	Mai's idea is correct	Mai's idea is not correct
Third	15.3	84.7
Fourth	10.6	89.4
Fifth	16.2	83.8
Sixth	7.8	92.2
Total	13.1	86.9

Table 4. The category of pupils' descriptions about the population density

A	By looking at the given figure. (*Because four children in the toy store are in the middle of the room.*)
B	Based on own experience in real life. (*Because people crowd around a popular amusement attraction at Disneyland also.*)
C	Based on multiplicative reasoning in the given figure. (*In the toy store, there are four children on two-mats, so population density is half-mat per one child. In the candy store, there are six children on six-mats, so it is one-mat per one child.*)
D	Based on feeling. (*It seems to be crowded as people spread evenly.*)
E	Based on additive reasoning about the number of mats in the given figure. (*In the toy store, there are six-mats with nobody on it. In the candy store, there are only two-mats with nobody on it.*)
F	Based on additive reasoning about the number of mats. (*If children sit on a mat one by one, there are four-mats with nobody on it in the toy store, but there are two-mats with nobody on it in the candy store.*)
G	Based on additive reasoning about the number of children. (*There are four children in the toy store, and there are six children in the candy store.*)
H	Based on primitive multiplicative reasoning. (*There are few children in the toy store than the candy store, and the area of both stores is the same.*)
I	Based on multiplicative reasoning. (*In the toy store, there are four children on eight-mats, so population density is two-mat per one child. In the candy store, there are six children on eight-mats, so it is four-thirds-mat per one child.*)
J	Based on equable reasoning. (*If children in the toy store spread evenly, then the candy store is crowded. Because there are few children in the toy store than the candy store, and the area of both stores is the same.*)
K	Other answer
L	No answer

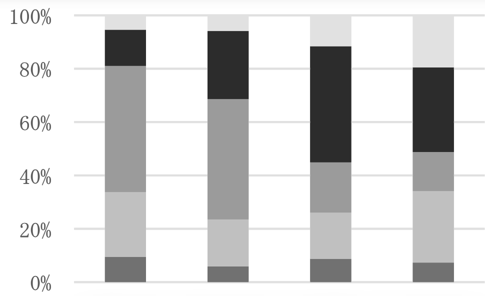
Table 5. Results of pupils' response categorized each type (Correct)

Category	The number of pupils	Percentage
A	19	5.8
B	9	2.8
C	5	1.5
K	10	3.1

Table 6. Results of pupils' response categorized each type (Not Correct)

Category	The number of pupils	Percentage
D	50	15.3
E	17	5.2
F	5	1.5
G	77	23.5
H	66	20.2
I	21	6.4
J	23	7.0
K	21	6.4
L	4	1.2

The results of pupils' responses as classified in Table 4 are displayed in Tables 5 and 6. In Tables 5 and 6, the proportion of *A*, *D*, *G*, *H* and *J* is high. The proportion of pupils' responses categorized are displayed in Figure 2 below.



	Third	Fourth	Fifth	Sixth
J	5.4	5.9	11.6	19.5
H	13.5	25.5	43.5	31.7
G	47.3	45.1	18.8	14.6
D	24.3	17.6	17.4	26.8
A	9.5	5.9	8.7	7.3

Figure 2. The proportion (in percentages) of pupils' response categorized as A, D, G, H, J

In Figure 2, one-half percent of pupils in the fifth and sixth grade are categorized as *H* and *J*. They have already learned a quantity per unit. Therefore, they can interpret a problem situation by using learned knowledge. On the other hand, there are pupils from fifth and sixth grade classes categorized as *A* and *D*. These pupils are unable to interpret a problem situation by using learned knowledge.

Discussion

In this section, the author identified the qualitative differences in models by using the theoretical framework (see *Table 1*). The pupils' descriptions were analyzed from the viewpoint as to how their models should be revised, because the solvers need to revise the models iteratively.

Pre-mathematical model

It corresponds to categories *A*, *B* and *D*. These models express the population density based on observation, own experience and feeling, so these are called "Sensitive Model." This model is not expressed by mathematical means, but could develop into a mathematical model. For example, if one interprets "*It is crowded when people are in the middle of the room*" as "*It is crowded when the distance between people is short,*" then one finds the value of the distance. The pupils who are categorized as *A*, *B* and *D* do not have a means of interpreting the situation mathematically, but they have the fundamental idea of a mathematical model.

Mathematical model

It corresponds to category *C*, *E*, *F*, *G*, *H* and *I*. The models of *E*, *F* and *G* express the population density focus on the difference in the numbers of mats or children. Since these models express the population density based on difference in one quantity, these are called "Additive Model." This model is useful if the areas of both stores are the same, but not if they are different. It is necessary to revise the model to be used in other situations.

The model of *H* expresses the population density focus on the numbers of mats and children. Since the model of *H* expresses the population density based on the numbers of mats and children, this is called "Primitive Multiplicative Model". This model is useful if the area of both stores or the numbers of children is same, but it isn't useful if the area of both and the numbers of children are different. So it is necessary to revise the model.

The models of *C* and *I* express the population density as a ratio that compares different kinds of quantities. Therefore these are called "Multiplicative Model." This model is premised on spreading children in the toy store evenly. It is important to reveal the premise in order to teach the idea that population density is a comparison or a ratio to someone who thinks that it is crowded when people are in the middle of the room (like Mai who did not think that it is necessary to spread children in the toy store). In view of this, it is necessary to revise the model.

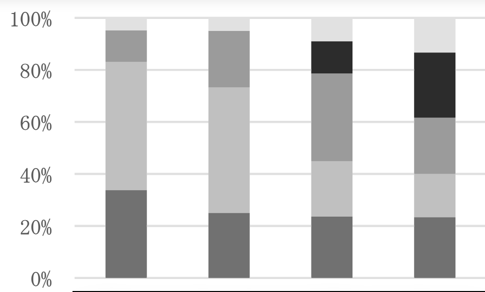
Meanwhile, the pupils who elicited the model of *C* changed the given conditions "there are four children in the toy store that is an eight-mat room" into "there are four children in the store that is a two-mat room." Generally speaking, children think that good mathematics problems come from good mathematics teachers and textbooks. The idea that children themselves can be a source of good mathematics problems has probably not occurred to many children (Kilpatrick, 1987, p. 123). However, the pupils mentioned above changed the problem. This change was not necessarily suitable, but their attitude toward mathematics was desirable. This showed that pupils in elementary school could change a problem.

The model of *J* expresses the population density focus to be spreading children in the toy store evenly. This is called “Equable Model.” The idea is to express population density as a ratio that compares different kinds of quantities. The expression of *J* was poor in style, but the pupils pointed out the premise.

The qualitative differences in pupils’ models are summarized in Table 7. The proportion of pupils’ responses is displayed in Figure 3. There are striking similarities between the interpretation of a problem involving multiple modeling cycles and the stages in the development of Piagetian conceptions of general proportional reasoning. The stages of the interpretation are described below: 1) qualitative reasoning, 2) additive reasoning, 3) primitive multiplicative reasoning, 4) pattern recognition, 5) general interpretation (Lesh & Doerr, 2003, pp. 19-21). The qualitative differences in pupils’ models have been constructed with reference to the stages of the interpretation by Lesh and Doerr (2003).

Table 7. The qualitative differences in pupils’ models

Level	Models		Category
1	Pre-Mathematical Model	Sensitive Model	*A, B, D*
2	Mathematical Model	Additive Model	*E, F, G*
3		Primitive Multiplicative Model	*H*
4		Multiplicative Model	*C, I*
5		Equable Model	*J*



	Third	Fourth	Fifth	Sixth
5	4.8	5.0	9.0	13.3
4	0	0	12.4	25.0
3	12.0	21.7	33.7	21.7
2	49.4	48.3	21.3	16.7
1	33.7	25.0	23.6	23.3

Figure 3. The proportion (in percentages) of level of the qualitative differences in pupils’ models

In Figure 3 the proportion of Level 1 and 2 decreases with advancing age while the proportion of Level 4 and 5 increases with advancing age. On the other hand, pupils from third and fourth grade can elicit the models of Level 4 and 5 though the proportion is low, i.e., they don’t learn a quantity per unit yet, but they can focus on two different kinds of quantities and provide the idea of spreading children in the toy store evenly.

Conclusion

This study intended to explore how pupils interpret a problem situation about population density. The author analyzed pupils' responses to them. Consequently, five different models were identified.

The results showed that pupils in elementary school could interpret the problem situation although previous studies on mathematical modeling have focused on educational stages beyond the middle school. The results depend on the problem of population density. That is, if problems include realistic aspects (e.g., "daily population density") and others' idea (e.g., Mai), then there is a possibility of bringing out pupils' interpretations.

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Mai Hirabayashi
University of Tsukuba
1-1-1 Tennodai, Tsukuba, Ibaraki 305-8577, Japan
hira@human.tsukuba.ac.jp