A study on students' conception of rate of change through *Yungbokhap* mathematics instruction

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Introduction

In this research, as part of design research to seek an instructional model to improve teaching and learning of Calculus, we investigated students' conception of rate of change and its transformation in a course of an integrative mathematics instruction. We distinguish "conception" from a "concept," so the term "a student's conception" means individual student's conceptual understanding of the concept of rate of change. Calculus that explains the order and laws inherent to various phenomena is a crucial tool to predict a changing situation. In particular, rate of change is a vital conceptual tool for students to explore change. Students, however, have difficulties in conceptually understanding rate of change and applying it mathematically to their real world situation. In this context, we adapted *Yungbokhap* mathematics instruction as a way to provide enriched learning environment for students to learn rate of change.

Yungbokhap mathematics instruction, an integrative and holistic approach to teaching and learning of mathematics, emphasizes autonomy, diversity, contextualization, and collaboration of students as instructional principles (Ju, Song, & Moon, 2013; Moon, 2014). These instructional principles reflect the characteristics of mathematics as human practice. Mathematics provides human beings with a conceptual framework for thinking, interpreting, creating and communicating about a real world problem. This implies that mathematics can be highly integrative and interdisciplinary knowledge. Furthermore, mathematics as human practice is created by active and dialogical inquiry of human beings contextualized within the need and the interest of an individual as well as of a community. Therefore, *Yungbokhap* mathematics instruction reflects the key characteristics of mathematics as practice and of a human as a learner.

In this perspective, we designed a teaching sequence based on the instructional principles of *Yungbokhap* mathematics instruction and implemented it to support students' meaningful understanding of rate of change. We collected data to investigate students' conceptions with regard to the rate of change emerging in class. Based on the analysis, this paper discusses implications for how to reform the curriculum of Calculus.

Teaching and learning of the rate of change

Rate of change is highly integrative in the regard that it has multiple meanings such as: geometric meaning as a slope of the tangent line, physical one as a concept of instantaneous speed, and algebraic one as a limit value of interval rate of change. However, students rarely understand these various meanings of rate of change. A number of studies reported that students have difficulty in coordinating the first derivative and the value of the limit to interpret the graph of the relevant function (Ubuz, 2007).

In this context, a variety of instructional methods was designed and implemented for the improvement of teaching and learning of Calculus. As a result of that, Hauger (1997) found that students used various resources to explain rate of change and that slope and change over intervals are powerful ways for students to think about rate of change. Also many mathematicians recommended the adoption of historical-genetic analysis, mathematical modeling and pragmatic problem solving (Doorman, 2005). Tall (1986, 1991) designed a course to help students grasp the idea of limit first intuitively through zooming, which he later formalized. Tall also tried the holistic approach that combines the numerical, graphic, and algebraic methods. Tall's approach is connected with the Calculus reform movement that began in the late 1980s (Tucker & Leitzel, 1995).

We reviewed various ways for teaching rate of change. The studies suggested the importance of students' own active engagement with an inquiry situated within an authentic mathematical context. Furthermore, they showed that teaching materials need to be based on a context of real world. In this regard, we thought that *Yungbokhap* mathematics instruction could provide an effective way to teach rate of change. From this perspective, this research added the key characteristics of the *Yungbokhap* mathematics instruction – an integrative and holistic approach to emphasize autonomy, diversity, context-centered, and collaboration – in order to improve teaching and learning of rate of change.

Methods of inquiry

Design experiment

Design experiment is directed at developing, implementing, analyzing and redesigning recursive practices to interact between the developed instruction and the theoretical insights (Ejersbo et al., 2008). From this methodological perspective, we designed an instructional program based on the following 5 principles. First, integration should be situated within students' lived world. Second, teachers should respect students' autonomy. Third, teachers should value and promote mathematical diversity in integration. Fourth, integration should be based on the dialogical relation among subject matter, teachers, students, and their lived world. Lastly, collaboration should be promoted in class based on integrative curricula.

Therefore, we constructed learning contents depending on the integrating approaches of different subject areas: multidisciplinary, interdisciplinary, and transdisciplinary approach. We used real world context (physical/social/global) as well as the mathematical context to help students contextualize. Also we formed the teaching and learning methods to enhance students' autonomy, diversity, and collaboration. To enhance students' conceptual understanding, we provided tasks that asked the students to explore the change of a graph in a variety of ways, which traces the historical-genetic trajectory of its conceptual development. Specifically, in GRC tasks, the students interpreted increasing and decreasing graphs over an entire domain. In IRC tasks, the students dealt with the slope of a graph on a certain interval. Also, in PwRC tasks, students explored the concept of limit in a neighborhood of a point. In FRC tasks, the students investigated the relation between function and its rate of change as a derivative function.

Research participants & research setting

The instructional program was implemented in the after-school math class for 5 weeks on Saturdays in a private high school in Seoul, Korea. Each session lasted 4 hours. Sixteen students participated in the class. Most participants ranked top in mathematics and had positive attitudes toward mathematics.

One of the researchers taught the class. She had two roles as a teacher and a researcher in this study. As a researcher, she adapted a theoretical model of *Yungbokhap* education to teach mathematics and designed the teaching sequence and the teaching materials for rate of change. As a mathematics teacher, she believed that *Yungbokhap* mathematics instruction would be beneficial to students because it provides learning environment to promote students' autonomy and diversity in the collaborative construction of mathematics. Thus, based on the instructional principles of *Yungbokhap* mathematics instruction, she orchestrated the practice of mathematics in class.

Data collection & data analysis

All the sessions were video-recorded. In particular, each group activity was video-recorded individually. The students' "conception map," which is similar to mind map, concerning rate of change and activity papers were collected. The conception map provided data of how students conceptualized the notion of change in mathematics before this class. After the class was completed, the researcher interviewed students in order to investigate students' conception of rate of change through their own voice and to identify students' learning experiences and the *Yungbokhap* instructional elements that impact on students' conceptual understanding in class. The interviews were recorded and transcribed for analysis.

Only five students were analyzed because they participated in every session without missing any. We call them as A, B, C, D, E for convenience. Among the five students, three had already learned Calculus in private institutions, and one previewed Calculus on her own. Except for one student, all participated in the integrating class of mathematics and science and programs in the gifted education institutions.

In the analysis, we first identified students' conception of rate of change into four categories: GRC (Global Rate of Change), IRC (Interval Rate of Change), PwRC (Point-wise Rate of Change), and FRC (Functional Rate of Change, which means derivative and objective representation of rate of change). Based on the order presented in the textbooks, we can regard that these categories have a hierarchy. We refined the categories by adding subcategories to get eight codes in total. The students' conceptions of rate of change were classified as shown in Table 1.

To analyze, we applied these subcategories to investigate how the students conceived the concept rate of change and how their conceptions had changed during class participation.

Table 1. The conception of rate of change framework

NAME	CODE	DESCRIPTION
Global Rate of Change	GRC1	A student cannot understand the meaning of slope. She is more interested in change than rate of change.
	GRC2	A student can recognize the rate of change.
Interval Rate of Change	IRC1	A student can see interval rate of change.
	IRC2	A student can explain point-wise rate of change using the concept of secant and its limit.
Point-wise Rate of Change	PwRC1	A student can see point-wise rate of change.
	PwRC2	In the explanation of derivative, PwRC2 can be seen.
Functional Rate of Change	FRC1	A student can distinguish between derivative and PwRC.
	FRC2	A student knows functional rate of change and does reversible and logical operations with it.

Characteristics of students' conception of "rate of change"

Individual students' conception

First, we analyzed the conception maps, which were drawn by the students in the beginning of the class. The result showed that two of the conception maps could be identified as a type of FRC1, two as IRC1, and one as GRC2.

Student A's conception could be classified as FRC1 because she tried to confirm her answers by using formulas after finishing each task. She rarely considered the context of a task, and usually tried to find a derivative directly using a formula in every task. Thus, FRC increased from GRC/IRC tasks to PwRC and it appeared most often in PwRC tasks. This implies that student A was familiar with the algebraic algorithm but had difficulties in qualitative approach through graphs.

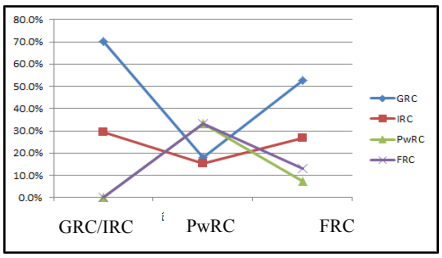

Figure 1. Student A's conception of rate of change

Student B also began with FRC1 and she led the discussion actively in her small group. In the case of student B, IRC appeared in 41.6% of the PwRC tasks as Figure 2 shows. The high proportion of IRC was due to her use of easy terms when explaining. Furthermore, she used IRC2 to explain the interval rate of change. Student B explained a concept of a tangent at a point on the curve using the concept of the extreme left and

the extreme right of the secant and this suggests that Student B understood the origin of a formula.

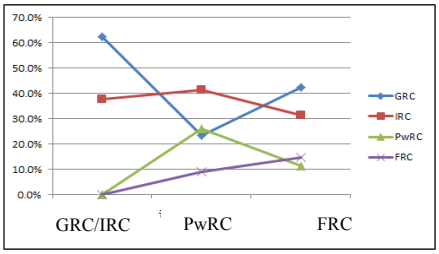

Figure 2. Student B's conception of rate of change

Student C with IRC1 was the only one who did not preview Calculus, but she participated in the small group discussion with some physical knowledge. It must be noted that the proportion of student C's FRC was the highest in the FRC tasks. We interpreted that student C tried to solve the problems by reading and interpreting the graph and activating her existing knowledge of various fields, especially that of science.

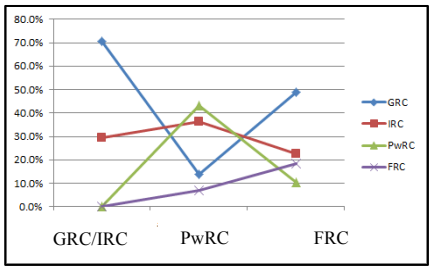

Figure 3. Student C's conception of rate of change

Student D with IRC1 guided the discussion in her small group and showed IRC1 with the highest proportion in the PwRC tasks like Student B as Figure 4 shows.

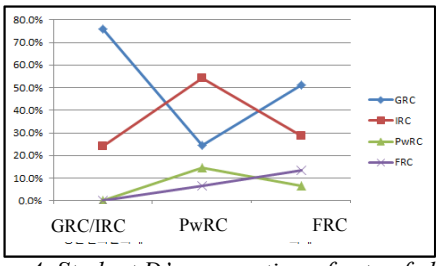

Figure 4. Student D's conception of rate of change

Student E with GRC2 could not discourse at all about FRC in GRC/IRC and PwRC tasks as shown in Figure 5. With the proportion of IRC being higher in the PwRC tasks and that of FRC in the FRC tasks, we interpreted these to be meaningful.

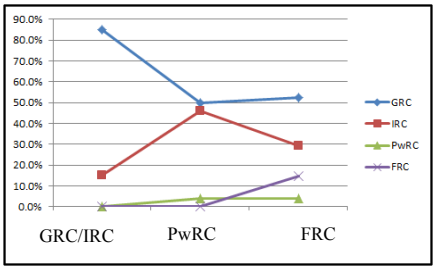

Figure 5. Student E's conception of rate of change

Conception in the context of diverse tasks

Figure 6 shows the frequency of discourse that appeared in the process of solving the GRC & IRC tasks. We knew that the five students reached GRC and IRC, which were the learning objectives of the tasks, although the frequencies of each type of conception were very different. In particular, students A, B, and C could understand that the slope of a tangent line is the limit of secant and this makes them tmove up to the point-wise rate of change beyond IRC1.

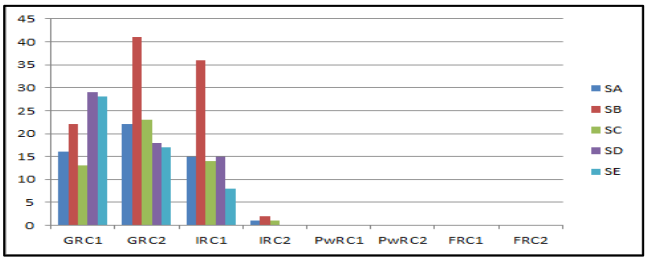

Figure 6. The conception of the rate of change in GRC & IRC Tasks

When we analyzed the students' discourse about the FRC tasks, all the five students acquired the FRC2 concept. When we compared students' first conception maps with the final ones, we found that every student improved their conceptual understanding of rate of change. As Figure 7 shows, the students almost understood the concept when a task is embedded in it.

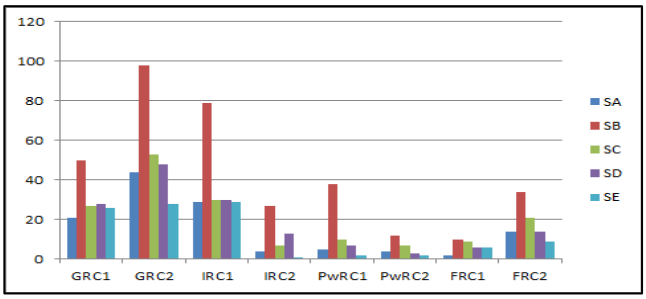

Figure 7. The conception of the rate of change in FRC Task

General Discussion

We discussed various aspects of the five students' conceptual understanding of the rate of change. Our analysis revealed several characteristics in the students' conceptual understanding of rate of change.

First, the students' conceptual understanding of rate of change changed over the course. Although the tasks did not exactly predict the type of a conception that the students adapted, they laid a ground for the emergence of new conceptions among students. However, it is important to note that the students' conceptual change was not linear but rather appeared to be spiral in the discourse. Although a student advanced to a higher level of conception, she often retreated to the previous conception when confronted with a challenging task. Secondly, diverse types of conceptual understanding of rate of change appeared in the students' discourses. Students used as many concepts as possible to solve a given task without being bounded to a certain type of conceptions. As mentioned above, IRC2 and PwRC2 showed the process from knowing interval and point-wise rate of change to approaching the next level of conception using the concept of limit. The frequency decreased over the course. This means that the students understood the slope of a tangent line at a point on the curve as the limit of the slope of secant lines and as a consequence, they understood more clearly the mathematical meaning of each type of rate of change.

Conclusion

This research was conducted as part of design research to develop an instructional model to provide an enriched environment for the teaching and learning of Calculus. Based on the analysis of the students' discourse in class, this research described the characteristics of the students' conceptions of rate of change that had emerged in the context of *Yungbokhap* mathematics instruction. We investigated the students' conceptions by administering eight tasks on rate of change and evaluating using two criteria: individual conceptions and conceptions related to the context of the tasks. Even though its emergence did not follow a linear trajectory among the various types of conception, the students' understanding of rate of change had advanced over the course.

The results of this research suggest that *Yungbokhap* mathematics instruction may provide an alternative model for teaching Calculus. In order to facilitate students' autonomy in learning, we adapted tools including CBR (Computer-Based Ranger), graphing calculator, and computers. These tools seem to enable students to reflect their preconception of change to develop their conceptual understanding of rate of change. Furthermore, in the *Yungbokhap* mathematics instruction, the students were encouraged to talk about mathematics in their own voices and listen actively to what their peers said. The students shared various types of conceptions of rate of change in small group discussions. It was in this dialogical relation that the students extended their conceptions by bridging the diverse perspectives on rate of change.

This research suggests that *Yungbokhap* mathematics instruction provides a setting for students to begin with their preconception of change to develop an understanding of rate of change as a functional relation. Grasping rate of change

as FRC is a critical step to understand the integration as the inverse operation of the differentiation, which forms the conceptual foundation for the fundamental principle of Calculus. Thus, further research is required to investigate how *Yungbokhap* mathematics instruction can be adapted to extend students' understanding of rate of change as FRC to learn the most central principle as well as the big ideas of Calculus.

References

Eves, H. (1990). *An Introduction to the History of Mathematics.* Saunders College Publishing.

Habre, S. & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *Journal of Mathematics Behavior, 25,* 57-72.

Hauger, G. S. (1997). *Growth of knowledge of rate in four pre-calculus students.* p.48. Washington DC: ERIC Clearinghouse microfiches. ED407230.

Ju, M. K., Song, R. J., & Moon, J. E. (2013). Inquiry into how to practice Yungbokhap education in mathematics: An analysis of focus group interview with school mathematics teachers. *Journal of Learner-Centered Curriculum and Instruction, 13*(4), 437-467.

Moon, J. E. (2014). *A study on students' concept of rate of change through integrated mathematics instruction*. Dissertation Ewha Womans University.

National Council for Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, DC: Mathematical Association of America.

Tall, D. (1986). *Building and testing a cognitive approach to the calculus using interactive computer graphics.* Ph. D. Thesis. The University of Warwick.

Tucker, A. & Leitzel, J. (1995). *Assessing calculus reform efforts: A report to the community.* Washington, DC: Mathematical Association of America.

Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: Stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology, 38*(5), 609-637.

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