Teaching proportional reasoning concepts and procedures using repetition with variation

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Introduction

According to a recent study on mathematics education, "The curriculum should afford sufficient time on task to ensure acquisition of conceptual and procedural knowledge of fractions and of proportional reasoning" (NMAP, 2008, p. xix).

> The curriculum should […] encompass instruction in tasks that tap the full gamut of conceptual and procedural knowledge, including […] solving problems involving ratios and proportion. (NMAP, 2008, p. 29)

Noche (2013) studied if supplemental self-paced instruction that focuses on the mastery of either concepts or procedures through repetition with variation helps young adults improve their performance in tasks designed to assess their proportional reasoning understanding and skills. We describe here the worksheets used in the study.

Proportional reasoning

We follow Vergnaud (1994) and define a rate as a quotient of two quantities in different measure spaces, a ratio as a quotient of two quantities within a single measure space, a product of measure as a product of two quantities in different measure spaces that is a quantity in a third measure space, and a (simple) proportion as an equality of two rates, or of two ratios, or of two products of measure. Lesh, Post, and Behr (1988) clarified Vergnaud's concept of measure space: "[T]wo measure spaces [are] considered different whenever they involve (1) a different set of objects, (2) a different type of underlying quantity (e.g., length, weight, time, etc.), (3) a different unit of measure, or (4) a different scheme for assigning numbers (i.e., measures) to objects in the space. For example, if students' heights are measured in inches and also in feet then these measurements will be considered to be in two distinct spaces [...]."

We use Karplus, Pulos, and Stage's (1983) definition of proportional reasoning as "reasoning in a system of two variables between which there exists a linear functional relationship," adding that this "leads to conclusions about a situation or phenomenon that can be characterized by a constant ratio" (p. 219). In Harel, Behr, Post, and Lesh's (1992) comparison of proportional reasoning tasks, one task-centered variable that they used is whether or not a task is numeric. A task is numeric if arithmetic computations are required in getting its solution; otherwise, it is nonnumeric. Nonnumeric tasks can be either number-free or quantified. For example, the task "Which is greater, 3/8 or 2/9?" is a quantified nonnumeric task; it can be solved without numerical computation: because $3 > 2$ and $8 < 9$, it follows that $3/8 > 2/9$. The task "Which is greater, 3/5 or 7/8?" is a numeric task.

Concepts and procedures

Noche's (2013) study operationally defined conceptual knowledge in proportional reasoning as knowledge of the multiplicative and additive mathematical principles of

Harel et al. (1992) and procedural knowledge in proportional reasoning as knowledge of correct arithmetic or algebraic procedures either to solve for an unknown in an equation describing a proportion or to find the order relation between two rates, ratios, or products of measure. Noche (2013) measured the amount of conceptual knowledge using nonnumeric tasks and the amount of procedural knowledge using numeric tasks. (This is consistent with how Rittle-Johnson and Siegler (1998, p. 100) described Dixon and Moore's (1996) study on proportional reasoning: "the first condition ['described without numerical specification'] assessed qualitative or conceptual understanding, and the second condition ['described numerically'] assessed quantitative or procedural understanding.")

Repetition with variation

The idea of repetition with variation is often seen in East Asian mathematics education. "With a set of practicing exercises that vary systematically, repeated practice may become an important "route to understanding [...]" (Leung, 2006, p. 43). Marton (as quoted in Leung (2006, p. 43)) refers to repetitive learning in the East Asian culture as "continuous practice with increasing variation." Marton, Wen, and Wong (2005, p. 312) found that Chinese college "students think that memorization springs from repetition and that understanding springs from variation."

The Western idea of rote drilling is not the same as the East Asian idea of repetition with variation. "Unlike when you read the same presentation of something several times in the same way and thus repeat the same thing again and again, when you read different presentations of the same thing or when you read the same presentation in different ways, something is repeated and something is varied" (Marton et al., 2005, p. 291).

Hiebert and Handa (2004) use the phrase "repetition with variation" to describe the "productive interplay between procedural and conceptual activity" that they found in a Hong Kong classroom. They suggest three ingredients that enable this: "a mathematical topic that is of an appropriate size and richness," "a carefully chosen sequence of tasks," and "basic definitions presented early in the lesson [that] provide an anchor."

We used the Kumon method (Ukai, 1994) as our model when we created the worksheets to follow the East Asian idea of repetition with variation.

The worksheets

Contents

Table 1 shows the descriptive titles of the eleven worksheets. Each has two versions: one with nonnumeric (conceptual) tasks, and another with numeric (procedural) tasks. Each is a booklet eight half-letter-sized pages long and includes a short discussion of the concepts or procedures involved, with examples. Each worksheet page has a label composed of an uppercase letter (A to F) identifying the worksheet, a lowercase a for the 'conceptual' worksheets and a lowercase b for the 'procedural' worksheets, and the page number (1 to 8) of that worksheet. The 22 worksheets are reproduced in Noche (2013).

The tasks vary in the physical principles underlying the problem situation and in the mathematical principles underlying the solution of tasks (Harel et al., 1992).

Table 1. Descriptive titles of worksheets

A	Locating numbers on a number line
B	Identifying points on a number line
C	Ratio and proportion using a linear scale
D	Ratio comparison problems
E	Mass of a liquid
F	Interlocking toothed gears
G	Sugar and water
H	Water rectangle
I	Masses of chocolate bar pieces
J	Volumes of liquids in different containers
K	Review of worksheets D, E, F, G, H, I, and J

The first three worksheets involve understanding and skills that may help in improving proportional reasoning. According to NMAP (2008, p. xix), "One key mechanism linking conceptual and procedural knowledge is the ability to represent fractions on a number line." Worksheets A and B cover number line representations, a representational support that has been shown to be effective (NMAP, 2008, p. 29).

Worksheet C uses Adjiage and Pluvinage's (2007) linear scale register (in particular, the double scale), which they consider as "a privileged tool for interpreting and processing ratio problems" (p. 157). The tasks in worksheets A, B, and C were adapted from those of Adjiage and Pluvinage (2007) who used them to teach ratio and proportion.

Worksheet D consists of ratio comparison tasks without context. Similar tasks appear, for example, in Adjiage and Pluvinage (2007) and Heller, Post, Behr, and Lesh (1990).

Worksheet E involves finding the mass of a liquid given its density and its volume. The tasks here were inspired by item HET1 of Adjiage and Pluvinage's (2007) post-test.

Worksheet F involves conservation of linear speed (the product of a gear's number of teeth and angular speed). The first author created these tasks but similar tasks appear, for example, in Kwon, Lawson, Chung, and Kim (2000).

Worksheet G involves dissolving grains in a liquid to get a solution with the same volume as the liquid. The first example in Ga1, the first item in Ga2, and the first item in Ga5 are items 1, 2, and 3, respectively, of McLaughlin's (2003) diagnostic test. The other tasks in worksheet G were adapted from these. Alatorre and Figueras (2004) used a similar task involving lemons and cups with sugared water.

Worksheet H involves comparing the volumes of liquid in identical containers but with different orientations. It was inspired by Kurtz's (1976, p. 34) water triangle apparatus, and is described further in Noche and Vistro-Yu (2015).

Worksheet I involves the decomposition and composition of a solid with uniform density. The tasks here were inspired by items PW03 and PW04 of Adjiage and Pluvinage's (2007) post-test.

Worksheet J involves conservation of volume (the product of a liquid's height and area in a container). The tasks here were inspired by items 9 and 10 of McLaughlin's (2003) diagnostic test, which were apparently adapted from the pouring water task of Suarez and Rhonheimer (as cited in Kwon et al., 2000, p. 1172).

Administration

The worksheets are to be answered individually without using books or calculators. One worksheet is to be done each day, taking around 15 to 30 minutes to complete (cf. Ukai, 1994, p. 91). Students answer the worksheets at their own pace, prioritizing performance over speed. Tasks are to be done in the order they are presented (cf. Ukai, 1994, p. 90, "highly sequential presentation"). Students may approach the teacher for short clarifications regarding the worksheets (cf. Ukai, 1994, p. 95, "The job of instructors").

At the start of each daily session, each student individually consults with the teacher who shows him or her how he or she performed in the previous worksheet. If there are few errors (10% or less), then the student corrects the errors with the help of the teacher, then answers the next worksheet alone. If there are many errors (more than 10%), then the teacher provides some brief feedback and the student repeats the whole worksheet alone. If the student takes much longer to finish a worksheet than the time allotted for it (twice the time or more), then he or she repeats the worksheet (cf. Ukai, 1994, p. 91, "prescribed time and mistake limits"). The time allotted for each worksheet is based on the average time taken by students in a pilot study (cf. Ukai, 1994, p. 104, "Standard Completion Time").

If the previous worksheet was done at home, then the student submits it at the start of the session and is given the next worksheet even though the submitted worksheet has not yet been checked (cf. Ukai, 1994, p. 91, "completed homework"). The teacher checks the submitted worksheets before the next session to determine what worksheets to assign during the next session (cf. Ukai, 1994, p. 91, "child's progress." Compare also p. 107, "grading and record-keeping").

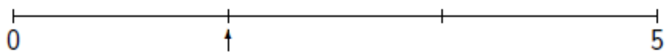
An example

Worksheet C is about ratio and proportion using a linear scale, and is to be done only after identifying points on a number line (worksheet B) has been mastered, which in turn is to be done only after locating numbers on a number line (worksheet A) has been mastered.

The first two pages of both versions of worksheet C introduce the concept of a double-scale number line. Terminology and notation are explained with examples and are followed by tasks that test understanding and skill. The examples and the tasks are arranged in a slowly increasing level of difficulty.

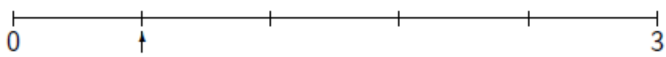
Ca3

Recall that for the single-scale number line shown below, the arrow points to the number $5/3$.



Similarly, for the double-scale number line shown below, the arrow points to the number $5/3$. This number line can be described by the relationship $5/3:1$.

For the single-scale number line shown below, the arrow points to the number $3/5$.



Similarly, for the double-scale number line shown below, the arrow points to the number $3/5$. This number line can be described by the relationship $1:3/5$.

Thus, $1:3/5$, $5/3:1$, and $5:3$ are some of the relationships that describe the double-scale number line shown below.

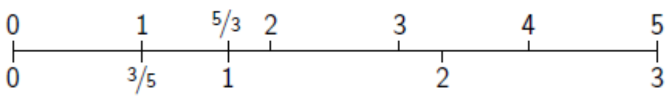


Figure 1. Ratio and proportion using a linear scale (conceptual instruction)

Ca4

For each double-scale number line shown, give the number that the arrow points to.

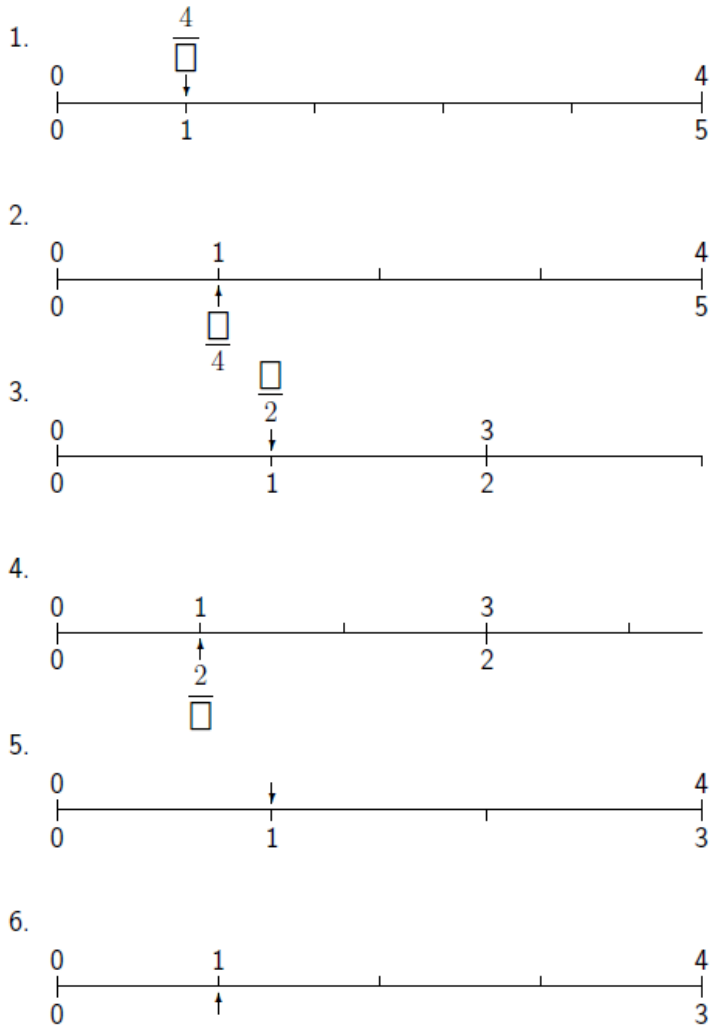
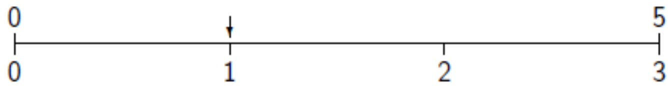


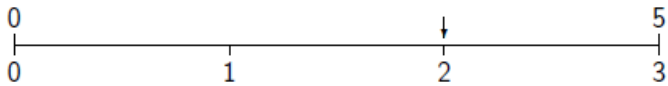
Figure 2. Ratio and proportion using a linear scale (non-numeric tasks)

Cb3

For the double-scale number line shown below, the arrow points to the number that is one-third of 5; it points to $\frac{1}{3}(5) = {}^{5}/_{3}$ or $1\ {}^{2}/_{3}$.

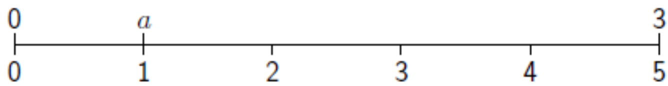


For the double-scale number line shown below, the arrow points to the number that is two-thirds of 5; it points to $\frac{2}{3}(5) = {}^{10}/_{3}$ or $3\ {}^{1}/_{3}$.



In general, if a double-scale number line is described by the relationships $a : b$ and $c : d$ (so that $a : b :: c : d$), then $a = \frac{b}{d}(c)$.

For the double-scale number line shown below, $a : 1 :: 3 : 5$, so $a = \frac{1}{5}(3) = {}^{3}/_{5}$.



For the double-scale number line shown below, the arrow points to $\frac{2}{5}(3) = {}^{6}/_{5}$ or $1\ {}^{1}/_{5}$.

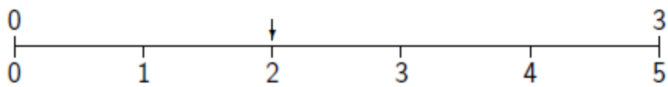
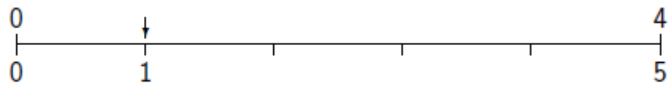


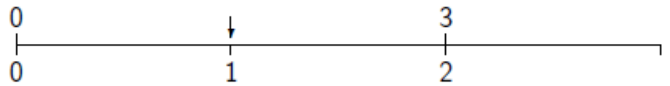
Figure 3. Ratio and proportion using a linear scale (procedural instruction)

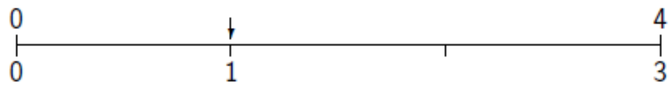
Cb4

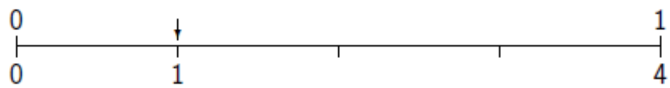
For each double-scale number line shown, find the number that the arrow points to. Show your solutions.

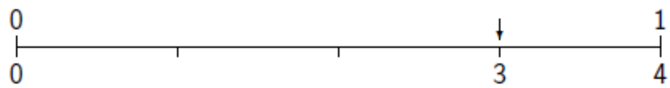
1. 

$\frac{1}{5}(4) =$

2. 

3. 

4. 

5. 

6. 

Figure 4. Ratio and proportion using a linear scale (numeric tasks)

Figures 1 and 2 show how a concept (the relationship describing a double-scale number line) can be used to find a missing quantity in a proportion. Although the tasks are quantified (involve numbers), they can be done without numerical computation.

Figures 3 and 4 show how a procedure (evaluation of an algebraic expression) can be used to find a missing quantity in a proportion. The use of numerical computation is emphasized by requiring the procedure to be written down.

Recommendations

Pilot testing of early versions of the worksheets revealed that, in general, students took longer to do the procedural version than the conceptual version. The worksheets were revised in an effort to make the completion times of the two versions more similar. Nevertheless, in Noche's (2013) study involving 46 undergraduate students (with an average age of 18 years and 5 months), the procedural version of the worksheets still took longer to complete on the average than the conceptual version.

The twelve students doing the procedural version of the worksheets took from 3 to 95 minutes (average of 24, median of 21) to finish each worksheet. The completion times for the ten students doing the conceptual version ranged from 2 to 60 minutes (average of 19, median of 15). A Kruskal-Wallis one-way analysis of variance by ranks shows that the times for the two groups differed significantly ($H = 13.69$, $p = .001$).

Noche's (2013) study used these worksheets to find empirical evidence on the causal relationships between conceptual and procedural knowledge in mathematics—how "[p]ossession of one type of knowledge is causally related to acquisition of the other" (Rittle-Johnson & Siegler, 1998, p. 78). It was thus important that the two sets of worksheets differ only in the type of instruction and not in the duration of instruction. The worksheets need to be further revised and tested to finally determine whether or not there is a significant difference in completion times between the two versions.

More significant improvements in understanding and skills may be obtained with a larger number of worksheets. Additional worksheets are now being planned.

Conclusion

These worksheets were created to investigate the causal relationships between conceptual and procedural knowledge in the domain of proportional reasoning using an East Asian perspective. They emphasize performance, speed, and continuous practice with tasks done in a strictly sequential order and repeated until mastery is attained. We hope that these worksheets provide a basis for future studies and more insight on instruction that uses repetition with variation.

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